VIBRATION PREVENTION IN ENGINEERING

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IN
ENGINEERING

BY

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PREFACE

THIS book is an outcome of and closely follows the outline of a series of lectures presented at the Harvard Engineering School during the season of 1930–1931.

The increasing importance of noise and vibration in the engineering work of the General Electric Company has brought to the author's attention many interesting problems, and has indicated a need for a convenient reference work on this subject. The purpose of this book is, therefore, to present in concise form a résumé of the various aspects of vibration prevention in engineering, which have come directly within the author's experience.

The book is intended to serve as a reference work for practicing engineers and particularly for the students in the Advanced Course in Engineering of the General Electric Company.

No one can do effective work in this field without some grasp of the fundamentals on which the subject is based. The method of presentation has, therefore, been to develop each topic in a way which leaves no questions as to the basic logic involved, but at the same time to omit long analytical developments not essential to this understanding.

Discrimination has been used in selecting and developing only those aspects of elasticity and dynamics which in the author's experience have been found to be particularly important in vibration work.

Although many fields have been touched upon, which could have been much further expanded, the use of material and viewpoints already conveniently available in other text books has, so far as possible, been avoided.

The scope of this volume comprises all the material necessary for handling effectively any usual vibration

problem. An effort has been made to enhance the value of the presentation by clear visualization of the theory and by directness of attack on practical problems.

The sections on the application of complex quantities to the solution of vibration problems represent methods developed and used by my associates E. H. Hull and W. C. Stewart, to whom I wish to express my indebtedness.

I also wish to express my appreciation of the assistance given me by Dr. B. L. Newkirk, and by Messrs. P. L. Alger, E. L. Thearle and A. L. Ruiz, who reviewed and criticized the manuscript.

I desire particularly to acknowledge the interest and vision of Prof. Arthur E. Norton, of the Harvard Engineering School, who was primarily responsible for the presentation of the lectures of which this book is the outcome.

A. L. KIMBALL.

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VIBRATION PREVENTION IN ENGINEERING

CHAPTER I

VIBRATORY MOTION IN GENERAL

1. Vibratory Motion.—Any motion which repeats itself through approximately equal intervals of time is called a periodic motion. Periodic motion is a very common form of motion in nature, as a little consideration will show. The planets perform periodic motions around the sun. Indeed, the rotation of the earth on its axis is a form of periodic motion. The swing of the pendulum, the reciprocating motion of a piston, or any motion classed under the head of vibrations is a periodic motion.

A vibration may be defined as a *periodic back-and-forth or reciprocating motion which reverses its direction twice every cycle.* Vibrations fall into two general classes: the first, a *free vibration* which has no driving force but which when once started repeats itself without the aid of any external agency, usually dying out only gradually; and the second, a *forced vibration* which is maintained by a driving force of any frequency.

2. Conditions Necessary for a Free Vibration.—In order that a given mass shall freely vibrate, a displacement from its *rest position* must be opposed by a so-called *restoring force.* When the body is displaced from its rest position, work is done against this force whose amount depends upon the amount of the displacement. When the body is set free, this potential energy is released by giving motion to the body so that when its rest position is reached the energy is all kinetic. If there are no energy losses during this motion the kinetic energy of motion, as the body passes

through its rest position, is equal to the potential energy imparted by the original displacement. This carries the body onward to a new displacement on the other side of its rest position until again all the vibrational energy becomes potential. This process will continue indefinitely, the vibrational energy alternating between potential and kinetic. In ordinary cases, such a freely vibrating body will gradually come to rest because of friction of different kinds which dissipates the vibrational energy. Such a vibration, where the body is acted upon by the restoring force alone, is called a *free vibration*. The frequency of a free vibration has one particular value called the *natural frequency* of that vibrating system.

The phenomenon of vibration decay, that is, the process of gradual falling off of amplitude, is characteristic of a free vibration, such, for instance, as the dying out of the swing of a pendulum.

3. **Forced Vibrations.**—Forced vibrations are of quite different character from free vibrations, and should be clearly differentiated from them.

A *forced vibration* is an oscillation of a body maintained by an external periodic driving force. The natural frequency depends upon mass and the restoring force and has only one value, for a simple system, whereas a forced vibration may have any frequency, depending on the frequency of the external stimulus including the natural frequency. In fact, a body with no restoring force whatsoever acting upon it will take up a forced vibration. The quiver of an unbalanced motor, always in step with its rotational frequency, is an example.

The natural frequency depends upon the physical properties of the system itself, having nothing to do with an external stimulus.

4. **Resonance.**—When the external stimulus which produces the forced vibration is exactly in step with the natural frequency, that is, the frequency of free vibration of the body, the amplitude of the vibration may become very

large indeed, and the vibration is said to be in *resonance* with the stimulus. In fact, in the ideal case where no friction is present, the amplitude of a resonant vibration will become infinite.

In actual cases, friction is always present, so that the resonant amplitude builds up until the rate of energy dissipation in friction becomes just as large as the rate of energy supply through the stimulus.

A resonant vibration may be thought of as a sustained free vibration. Although sustained by an external stimulus, it is not forced in the sense that a frequency other than the natural frequency is maintained. If the stimulus were removed, the vibration would continue at the same frequency with slowly dying amplitude.

An example of resonance of electrical oscillations familiar to all is found in the tuning of radio sets. The natural frequency of the electric circuit in the receiving set is made to correspond with the frequency of the oncoming electric waves, so that large resonant electrical amplitudes are built up.

A powerful station in close proximity will produce forced vibrations of sufficient intensity to be audible without tuning being resorted to.

An alternating electric current is an example of a forced electrical oscillation. The resonant condition is dangerous because of local high voltages, and is avoided.

5. **Vibrational Energy.**—A body vibrating freely at a constant amplitude is a system of constant energy, as no energy is received from or given up to the outside. Thus, during the periodic cycle of transfer of energy between potential and kinetic the sum of these two must always be a constant. In fact, this knowledge affords an important means of determining free vibration frequencies to be discussed later. The potential energy at the rest positions of the body at the ends of its swings must just equal the kinetic energy at the instant of its maximum velocity as it passes through its position of zero displacement.

Such a vibrating system with no dissipation present is an example of a *conservative system* of forces, all the energy being conserved at every instant within the system, regardless of how it moves.

6. Elasticity and Vibrations.—By far the most common source of potential energy in a free vibration is the potential energy of elastic deflection, and a consequent elastic restoring force. In this treatment, elastic restoring forces will be dealt with almost exclusively.

Another common source of potential energy is that due to gravitation, the restoring force being gravitational. A gravity pendulum is an illustration in point. Many vibrations are produced by restoring forces, both elastic and gravitational in character, such as a gravity pendulum whose frequency is increased by a spring at the top. In elastic vibrations, particularly when the frequency is high, the gravitational force is negligible and so is not considered.

Centrifugal force may furnish a restoring action, as illustrated by the increase in stiffness of long turbine buckets, due to this cause, in a turbine wheel revolving at a high speed. If the buckets are short and stiff, this effect is small and may be neglected in comparison with the elastic restoring force.

Common examples of elastic restoring forces are the deflection of springs of different materials, the bending of bars, or the compression and extension of bars, the compression of a gas or liquid. In fact, any conceivable elastic distortion of any elastic material may furnish the restoring force for some type of vibration.

7. Vibrations and the Principle of Minimum Energy.—One of the broadest general principles in mechanics is the *principle of minimum potential energy*, whereby any system of bodies or particles, however they may be connected, tends to assume a shape such that its potential energy is a minimum. A ball rolls down to a position of minimum gravitational energy. A compressed spring tends to expand, until its potential energy is a minimum. Two

springs compressed against each other adjust their deflections until their resultant potential energy is a minimum, etc., regardless of the complexity of the system. A vibration is the result of the effort of an elastic system to seek its position of minimum energy. It acquires a velocity in this effort which carries it beyond the position of minimum energy, and it then seeks it again and then again, through many repeating cycles, if the friction is small. The position of minimum energy is evidently identical with the rest position as previously defined. Indeed, any elastic system in stable equilibrium must have the capacity to vibrate back and forth through its position of stability if the friction is small. The test of stability is minimum potential energy such that any small displacement of the system from this position causes the energy to rise. It is for this reason that the capacity to vibrate is such a universal characteristic of material bodies in nature.

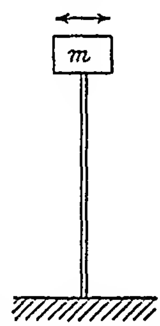


FIG. 1. — Mass on Reed.

8. Degrees of Freedom.—The number of degrees of freedom of a system capable of vibration is measured by the number of independent free vibrations, each with its own characteristic frequency, which that system may assume.

The weightless reed of Fig. 1 carrying the mass *m* has (omitting possible torsional oscillations), two degrees of freedom, one in the plane of the paper and one of higher frequency where the mass vibrates in a plane perpendicular to the plane of the paper with an edgewise deflection of the reed. If the reed is broad and thin the latter frequency is not readily excited, and we have practically a system of one degree of freedom.

The mass on the weightless helical spring of Fig. 2 also represents a system of one degree of freedom if the sideways motion is prevented by the pair of rods on which it slides. Without the rods it may assume a pendulum swing in either one of two directions perpendicular to each other

with a restoring force due to gravity, as well as a torsional oscillation about any one of three perpendicular axes, giving it six degrees of freedom in all.

A cylindrical mass supported on three springs as shown in Fig. 3 also has six degrees of freedom, four of which are easily excited, namely, a sideways swing, an up-and-down bounce, a torsional frequency about a vertical axis, and a sideways shearing oscillation. These four inde-

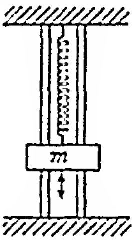


FIG. 2.—One Degree of Freedom.

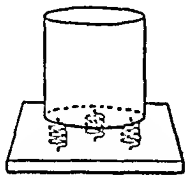


FIG. 3.—Six Degrees of Freedom.

pendent frequencies can be readily observed in an experimental model.

From a mathematical standpoint the number of degrees of freedom of a system is measured by the number of *independent* space coordinates by which its motion must be specified.

A continuous elastic body like a long deflected rod may vibrate in any of an infinite number of frequencies, and thus represents a system of an infinite number of degrees of freedom. This is characteristic of any continuous elastic body, assuming its molecular structure to be of infinitely small dimensions. From a practical standpoint, however, only the lower frequencies are easily excited. These cases will be subsequently discussed in detail.

CHAPTER II

SIMPLE LINEAR VIBRATION; HOOKE'S LAW AND VIBRATIONS; ELASTIC CONSTANTS

9. Simple Linear Vibration.—Take the case of a weight W vibrating up and down on the end of a weightless spring. The frequency is given by the formula

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where k = elastic constant of the spring

= force per unit deflection.

If a downward deflection d is produced by the weight W, evidently

$$k = \frac{W}{d}, \text{ so that (1) reduces to}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{d}} = \frac{3.13}{\sqrt{d}} \text{ cycles per second} \quad . \; . \; . \quad (2)$$

where g = 386 inches per second squared, and d is measured in inches.

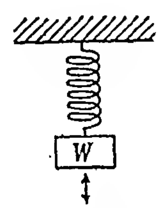


FIG. 4.—Simple Linear Vibration.

Thus the frequency of a system like that of Fig. 4 can be at once calculated when the gravity deflection d is known. Consider the case of a heavy motor on a comparatively light structure. By measuring the small downward deflection of this structure produced by the motor, the up-and-down natural frequency of the system is quickly estimated by formula (2). Cases often arise in engineering where this formula is applicable.

10. Derivation of Formula for a Linear Vibration.—Formula (1) is easily derived by an application of Newton's second law of motion, $f = ma$ where f = force and a = acceleration. If the mass m is given a displacement y, from its rest position, the restoring force = $-ky$ from the definition of k. It is negative because the force is directed towards the origin in the negative y direction. Thus

$$f = ma = m\frac{d^2y}{dt^2} = -ky$$

or

$$m\frac{d^2y}{dt^2} + ky = 0. \qquad (3)$$

If $\quad y = y_0 \quad$ and $\quad \frac{dy}{dt} = 0 \quad$ when $\quad t = 0,$

the solution of this differential equation is

$$y = y_0 \cos \omega_c t \qquad (4)$$

where $\omega_c = \sqrt{\frac{k}{m}}$ and y_0 = amplitude of y.

Since $\quad \omega_c = 2\pi f_c = \sqrt{\frac{k}{m}}, f_c = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}}.$

The symbol f_c is used to denote the free vibration frequency as distinct from any other frequency as might arise in the case of a forced or sustained vibration.

It is seen in Fig. 5 that the solution of equation (3) as given by (4) is represented graphically by the projection of a vector of length y_0 on the horizontal diameter of the circle described by the tip of this vector as it revolves at the constant angular velocity ω. The important point to note here is that this is *simple harmonic motion*, as simple harmonic motion is defined in just this way.

11. Hooke's Law and Harmonic Motion.—In the derivation of equation (3) the restoring force was taken as $-ky$. This assumes that Hooke's Law is obeyed, that stress is directly proportional to strain, or force is directly propor-

tional to displacement. Thus, whenever the restoring force is directly proportional to displacement, the resulting vibration is simple harmonic, or just harmonic as it is usually called. Since Hooke's Law that stress is directly proportional to strain is to a close approximation universally true of practically all the elastic materials of nature, vibrations involving elastic deflections are also universally harmonic in character.* This is fortunate, as a harmonic vibration is the simplest possible type of periodic motion from several standpoints. It is easily handled mathematically. Its frequency remains the same regardless of amplitude. In fact, the vibrations of all musical instruments must be closely governed by Hooke's Law; otherwise the individual notes would change with the intensity of the vibration, which common observation tells us is not true.

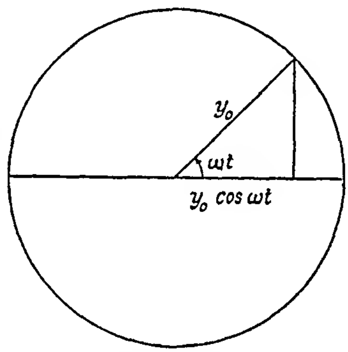


Fig. 5.—Simple Harmonic Motion.

12. Superposition of Vibrations; Law of Similar Systems.—A universal law of harmonic vibrations is that for systems of two or more degrees of freedom, including continuous elastic media with an indefinite number of degrees of freedom, any number of the characteristic free vibrations may be superimposed upon each other without affecting each other either as to their individual frequencies or amplitudes. For any one characteristic free vibration of a system of several degrees of freedom, all component parts vibrate exactly together, that is, they all have their maximum velocities at the same instant and all come to rest at the same instant. The phase of some parts may be reversed with respect to other parts, however, but this is always an exact 180° reversal, so that all parts vibrate in step. The portions of the system which are reversed in phase are separated from those which

* For a rigorous mathematical discussion of this point see reference 1.

are unreversed by stationary points called *nodes*. The irregular appearance of the vibration of such a system is due to two or more of these characteristic vibrations taking place on top of each other, independently. Take as an illustration the simple case of a system of two degrees of freedom shown in Fig. 6 (considering up-and-down vibrations only).

The two modes of vibration of this system are easily excited independently. For the lowest or gravest frequency, all parts oscillate up and down together, with the maximum amplitude at the bottom and a node at the top at the point of attachment. For the second or higher mode of vibration all parts vibrate in step as before, but m_1 and m_2 vibrate towards and away from each other in opposite phase with a node in the spring between them, and also a node at the top as before. All parts of the system between these nodes vibrate in opposite phase to those below the lower node. If the system is given a downward impulse, both vibrations take place at once, which, with keen observation, can be separated by the eye.

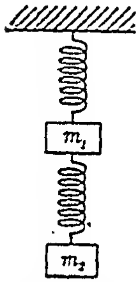


FIG. 6. — Two Degrees of Freedom.

The possibility of superimposing vibrations in this way follows from Hooke's Law. The stresses produced by one system of displacements add directly to those produced by a second system of displacements, each system producing its own stress components regardless of the presence of the other. In other words, the stress components produced by one mode of vibration at any instant are the same as if the other mode of vibration were absent. The same reasoning can be extended to systems having any number of modes of vibration.

A very interesting law, called the *law of similar systems*, can be shown to be true, on the basis of the laws of mechanics and principles of elasticity just given. This may be stated as follows: *When two vibrating systems are made of the same material and are exactly similar in dimensions,*

though not of the same size, their frequencies of vibration are inversely proportional to their linear dimensions.

13. Harmonic Analysis.—It was first shown by the French mathematician Fourier that any wave shape whatsoever can be exactly represented by the so-called harmonic series

$$y = y_0 + y_1 \cos(\omega t + \theta_1) + y_2 \cos(2\omega t + \theta_2) + y_3 \cos(3\omega t + \theta_3) \text{ etc.} \quad (5)$$

It is necessary only to chose the amplitudes y_1, y_2, etc., and the phase angles θ_1, θ_2, etc., correctly. The individual terms in this series are called harmonics, because each by itself represents a simple harmonic motion. The lowest frequency term, containing ω, is called the first harmonic or fundamental; the second one, containing 2ω and of twice the frequency of the fundamental, is called the second harmonic; the next the third harmonic, etc. The simple system of Fig. 4 has no higher harmonics, only the fundamental being present. The motion of any point of the system of Fig. 6 contains two frequencies.

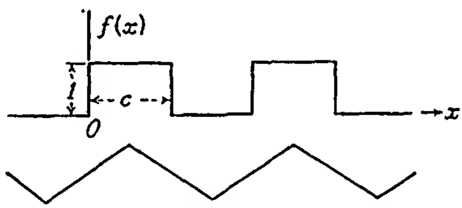


FIG. 7.—Periodic Curves.

The periodic curves as shown in Fig. 7 are readily represented by a series of the type (5), an infinite number of harmonics being required to obtain the mathematically exact curve. The series for the first of these is as follows:

$$f(x) = \frac{4}{\pi}\left\{\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \dots\right\}$$

If a few terms of this series are plotted against x as each successive curve thus obtained is added, the rectangular shape of the wave will be seen to be more and more nearly approximated.

If a piano wire is struck a sharp sudden blow with the hammer, a wave with a relatively sharp peak is started and travels back and forth in both directions. This wave contains a very large number of harmonics, the upper of which, however, soon die out, leaving the fundamental and lower harmonics as predominent, which give the tone its quality.

14. Young's Modulus, Stress and Strain.—Doubtless the first of the elastic constants to be defined was Young's modulus, usually designated by the letter E. The reason for this is that it is comparatively easily evaluated from such experiments as the force required to produce a certain elongation in a given wire or bar, being defined as the force per unit area of cross-section divided by the elongation per unit length produced in the stressed specimen.

If force per unit area $= F/A = \sigma$, called the stress, and if the elongation of the bar $= u = \epsilon l$, where $\epsilon =$ elongation per unit length, called the strain, and $l =$ length,

$$E = \frac{F}{A} \div \frac{u}{l} = \frac{\sigma}{\epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In other words, E is numerically equal to the stress necessary to make $u = l$, or that required to double the length of the specimen, assuming that Hooke's Law that stress is proportional to strain, is followed up to that point.

The relation (6) which may be put in the form $E\epsilon = \sigma$ will be often referred to from now on, where σ is called the tensile or compressional stress and ϵ the strain.

The stress σ may be thought of as the load on the faces of a unit cube of the stressed material perpendicular to the direction of the force, and ϵ as the amount that the unit cube is elongated or compressed.

15. Shear Modulus.—If, besides Young's modulus, the shear or rigidity modulus, designated by N, is known, the elastic properties of an isotropic material are completely specified. An isotropic material is one whose physical properties are the same in every direction.

Not only does a bar of material have the power to resist tension and compression, but it can also withstand torsion. Torsion sets up a stress which tends to produce a slipping or shearing of successive sections of the bar across each other. Shearing stress is defined as the stress per unit area which acts parallel to a surface instead of normal to it, and which tends to produce a slipping of successive layers. This causes a distortion of the material called a shearing strain.

Figure 8 shows a unit cube whose faces perpendicular to the paper are acted upon by a shearing stress. Note that if one pair of opposite faces are so stressed, the other pair must be stressed in shear by the same amount in order to maintain equilibrium. It is a general law of elasticity that perpendicular planes at any point in a material under stress must always have such equal balancing shearing stress components. If the shear on one plane is zero in all directions, it is zero in all perpendicular planes; or if zero in one direction it is zero in the corresponding perpendicular plane.

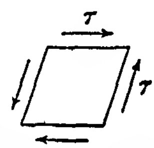


FIG. 8.—Shear on Unit Cube.

Figure 8 shows the type of strain produced by a shearing stress on a unit cube. Its faces become slightly diamond shaped.

The shear modulus N is defined in a way exactly analogous to Young's modulus as given by equation (6), that is

$$N = \frac{\tau}{\epsilon_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where τ is the shearing stress or load per unit area, and ϵ_s is the amount that the upper surface of the unit cube has slipped or sheared with respect to the lower surface. Note that ϵ_s is also equal to the angle of tilt of the vertical faces of Fig. 8 from the perpendicular.

16. Poisson's Ratio.—If a bar of elastic material is stretched or compressed by a small amount its diameter

does not remain constant, but decreases slightly with tension and increases with compression. The ratio of this sideways contraction or expansion of a unit cube to the corresponding stretch or compression is a constant for the material; it is called Poisson's ratio, and it will be signified by the Greek letter ν. Poisson's ratio has a value which varies from $\frac{1}{2}$ to $\frac{1}{3}$ for ordinary elastic materials. It is never greater than $\frac{1}{2}$, otherwise the volume of the material would decrease with tension and increase with compression, which is impossible for an isotropic material. If elastic deformation is attended by no volume change, Poisson's ratio has the value of almost exactly $\frac{1}{2}$. This condition is very nearly fulfilled by rubber and gelatin. Although E and N for these materials are small, they are very resistant to volume compression. An example of a material with Poisson's ratio almost equal to zero is cork. A cylinder of cork can be compressed along its axis by a large amount and show practically no change of diameter. Its elastic properties are rather poor, however, in that it does not immediately recover its original shape when the load is removed.

The following table gives approximate values of Poisson's ratio for a number of materials:

TABLE I

MATERIAL	POISSON'S RATIO
Iron	0.28
Steel	0.28–0.31
Copper	0.35
Brass	0.33
Glass	0.24
Rubber	0.50
Cork	0.00

17. Relation between Elastic Constants.—The three elastic constants just defined are Young's modulus E, the rigidity modulus N, and Poisson's ration ν. Any two of

these constants completely determine the elastic properties of a material, so there must be a definite relation connecting them. This relation may be found as follows:

Let $ABCD$ be the side of a unit cube before it is strained. Assume a tension σ to be applied at the top and bottom faces so that the original shape $ABCD$ of the side shown in Fig. 9 is distorted to the shape $A'B'C'D$. The tensile strain produced by σ is $AA' = \epsilon$. Poisson's ratio is evidently $C'C \div AA' = \nu$. It is seen from the figure that, although σ produces no tangential stress on the top plane AB, it does produce a tangential or shearing stress across any plane inclined to AB. This shearing stress is a maximum on a plane tipped 45° from the horizontal, such as AC. It is understood that AC is the edge of a plane perpendicular to the plane of the paper. The component of σ which is parallel to AC is evidently $\sigma \div \sqrt{2}$. Since plane AC has an area $\sqrt{2}$ times the unit area AB, the shear *per unit area* on AC produced by σ is $\sigma/2$, which is called τ (§ 15).

Therefore

$$\text{Shear on } AC = \tau = \frac{\sigma}{2}. \quad . \quad (8)$$

For clearness, the strains are shown greatly exaggerated in Fig. 9. Actually, they are so small that the change of areas and consequent change of stress intensities thus produced may be neglected. Such corrections are called second-order corrections and are universally neglected in all the mathematical theory of elasticity. It is only when the strains are so large that the shape of the body changes materially, as for a material like rubber, that this approximation results in appreciable errors. Thus, in estimating the shearing strain produced by the stress τ across plane

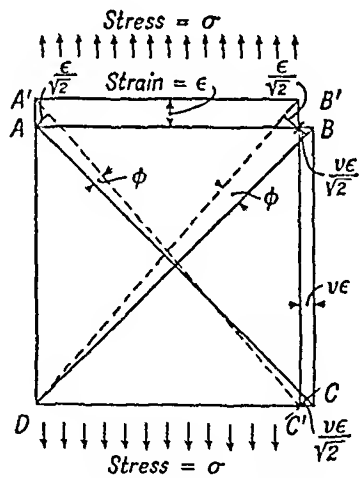


FIG. 9.—Strained Unit Cube.

AC, the final length $A'C'$ is taken equal to the original length AC, etc.

From the figure, $\phi = \left(\frac{\epsilon}{\sqrt{2}} + \frac{\nu\epsilon}{\sqrt{2}}\right) \div \sqrt{2} = \frac{\epsilon + \nu\epsilon}{2}$.

But the shearing strain ϵ_s with respect to AC as defined in § 15 is equal to 2ϕ, so that

$$2\phi = \epsilon + \nu\epsilon = \epsilon_s \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where ϵ = tensile strain and ν = Poisson's ratio.

Using (6), (7), and (8)

$$\tau = \frac{\sigma}{2} = \frac{E\epsilon}{2} = N\epsilon_s \quad \text{or} \quad E\epsilon = 2N\epsilon_s \quad . \quad (10)$$

Substituting (9) in (10), $E\epsilon = 2N\epsilon\,(1 + \nu)$, or finally

$$E = 2N\,(1 + \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is the required relation connecting the three elastic constants of a material.

Although here proved for one special case it can be shown to be generally true. (See reference 2.)

18. Further Remarks on Elasticity.—It is seen that a single tension or compression always produces shearing stress in a material. The only possible type of stress in which shear is absent is found when the compression or tensile stress is the same in every direction, as is the case at any point in a fluid under pressure. It is evident that a fluid cannnot resist a shearing force. The slightest shear on a fluid causes flow, due to the slipping over each other of adjacent layers. Thus, the shear modulus of a fluid is zero. Young's modulus also has no meaning, in the case of a fluid, since, as just explained, a stress in one direction must cause shearing forces which in turn cause flow. Thus the only elastic modulus possible for a fluid is the compressibility modulus or bulk modulus which is a measure of its resistance to hydrostatic pressure. The bulk modulus of air is small, but that of water is compara-

tively high. Of course, a solid also has a bulk modulus, but this is readily expressed in terms of any two of the three elastic constants previously defined.

For further explanations of these quantities and how they are derived, works on elasticity should be consulted. (See reference 2.) The purpose of the discussion of elasticity just given is to furnish a basis for the discussion of vibrations which follows. Nearly all vibrations involve elastic deformations, so that an appreciation of some of the basic laws of elasticity as given is essential to the study of vibrations.

CHAPTER III

VIBRATION OF LOADED CANTILEVER; BENDING MOMENT AND POTENTIAL ENERGY AT SECTION OF DEFLECTED BAR

19. Vibration of Weightless Cantilever and Beam with Load.—If a weight is placed on the end of a cantilever as shown in Fig. 10, the weight being several times more massive than the cantilever, the system may be treated as one of a single degree of freedom and the frequency found to a first approximation by neglecting the weight of the cantilever.

The simplest way to estimate the frequency is to measure the gravity deflection d in inches produced by the weight, measuring this in a line opposite the center of gravity of the weight.

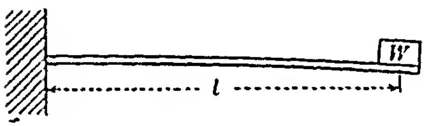


FIG. 10—Cantilever with Load.

The frequency is then readily calculated by formula (2), that is, $f = 3.13/\sqrt{d}$ cycles per second.

If d is not readily measured, the formula for d as produced by a weight W placed on the end of a cantilever may be used. This is $d = Wl^3/3EI$, where l = length of the cantilever to the center of gravity of the weight, E is Young's modulus for the cantilever material, and I the moment of inertia of its area of cross-section, sometimes called the second moment of this area. If the cross-section is rectangular, $I = bh^3/12$, where b = breadth and h = thickness or depth. The value of d thus calculated may then be used in place of the measured value.

If the elastic member is a beam supported by pivots at both ends, which carries the weight at its middle point,

instead of a cantilever, the process is just the same except that for such a beam $d = Wl^3/48EI$. The method can evidently be extended to beams supported in other ways by using the right expression for d. (d = deflection at center of gravity of the weight.)

It is evident that neglecting the weight of the beam leaves out of account some of the inertia of the vibrating system, so that the frequencies so calculated come out a few per cent too high.

20. Elastic Constant of Cantilever and Beam.—The deflection d at the end of a cantilever is found from the general formula for bending moment in a bar

$$M = EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

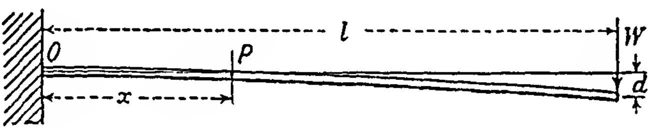


FIG. 11.—Uniform Cantilever.

where M = bending moment, I = the second moment of the area of cross-section, and d^2y/dx^2 is the curvature of the bar due to elastic deflection at the section under consideration. The complete derivation of this important formula will be given in a later section. Consider the cantilever of uniform cross-section shown in Fig. 11 carrying a weight W on its end. Let y be the downward deflection from the horizontal at any point x. The bending moment at x due to $W = W(l - x)$.

Thus

$$M = EI\frac{d^2y}{dx^2} = W(l - x) \quad . \quad . \quad . \quad (13)$$

Integrating:

$$EI\frac{dy}{dx} = w\left(lx - \frac{x^2}{2}\right) + c_1$$

When $dy/dx = 0$, $x = 0$, so that $c_1 = 0$.

Integrating again:

$$EIy = W\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) + c_2$$

When $y = 0$, $x = 0$, so that $c_2 = 0$. Thus:

$$y = \frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If

$$x = l, \quad y = d = \frac{Wl^3}{3EI} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This is seen to check the expression for d given in § 19.

The elastic constant

$$k = \frac{W}{d} = \frac{3EI}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If this is substituted in (1)

$$f = \frac{1}{2\pi}\sqrt{\frac{3EIg}{Wl^3}} = 3.13\sqrt{\frac{3EI}{Wl^3}} \quad . \quad . \quad (17)$$

The same result would of course be obtained by substituting (15) in (2) as indicated in § 19.

The calculation of the elastic constant for a beam of uniform cross-section supported and loaded in other ways than that given can be calculated by the method just given.

If the cross-section is not uniform, I is not a constant but depends upon x, which must be taken account of. The problem is of course more complicated in such a case.

21. Derivation of Bending Moment Formula.—The formula

$$M = EI\frac{d^2y}{dx^2}$$

just used is derived as follows.

Figure 12 shows a cross-section of a bar subject to bending. Let the x axis coincide with the neutral axis,

and the upper half be in tension and the lower in compression.

It will first be shown that the neutral axis must pass through the center of gravity of this area.

For equilibrium the total tension where y is positive must equal the compression where y is negative.

Expressed mathematically,

$$\int\int \sigma dxdy = 0. \quad . \quad . \quad . \quad . \quad (18)$$

where σ is the stress intensity, and the integration covers the entire area of cross-section.

Consider a section of the rod of unit length, bent from the straight to a curve of radius of curvature as shown in Fig. 13. In actual cases, ρ is very great compared with the diameter of the rod.

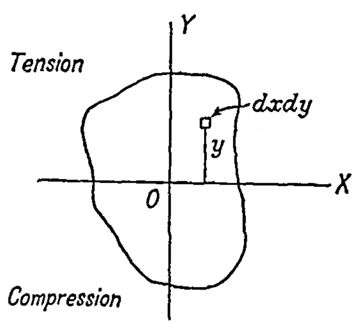


FIG. 12.—Cross-Section of Bar.

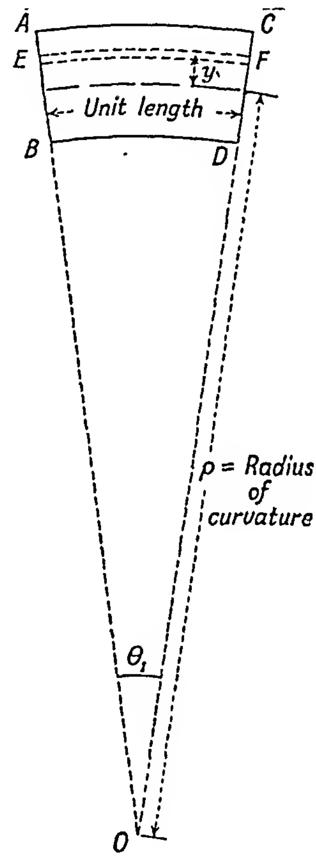


FIG. 13.—Deflected Section of Rod.

Consider a fiber EF at a distance y from the neutral axis as shown in Fig. 13. The elongation of this fiber is equal to the strain ϵ, since the section is of unit length.

If θ_1 = the angle made by AB and CD extended,

$$\epsilon = y\theta_1 = \frac{y}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The angle θ_1, which is seen to be the angle of curvature per unit length, is called simply the curvature; its reciprocal is equal to ρ, the radius of this curvature. In an actual unit length, ρ may change somewhat, as the bending moment varies along the unit length. Here we have assumed a slice of infinitesimal thickness to be expanded to unit length, all of which carries the same bending moment and thus has the same radius of curvature ρ.

From equation (6),

$$\sigma = E\epsilon$$

From equation (19)

$$y = \rho\epsilon = \frac{\rho\sigma}{E}$$

Therefore

$$\sigma = \frac{Ey}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Substituting (20) in (18),

$$\frac{E}{\rho}\int\int ydxdy = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

from which it is seen that the x axis, taken as the neutral axis, passes through the center of gravity or centroid of the area of cross-section. By definition, a line must pass through the center of gravity of an area when the sum of every element multiplied by its perpendicular distance to that line on one side of that line equals the sum of the same quantities taken on the other side of the line, which condition is satisfied by (21).

From Fig. 12 the total moment about the neutral axis is seen to be

$$M = \int\int \sigma ydxdy = \int\int \frac{Ey^2}{\rho}dxdy = \frac{EI}{\rho} \quad . \quad (22)$$

where $I = \int\int y^2 dxdy$ integrated over the whole section. I is recognized to be the second moment of cross-section of the rod (§ 19). It is a geometrical property of the cross-section of the rod and has no physical correspondence

to the moment of inertia of a body usually specified by the same symbol.

From calculus it is shown that

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \text{ which equals } \frac{d^2y}{dx^2}$$

when dy/dx is small as it is in deflecting beams when x is measured along the beam. Substituting this in (22):

$$M = EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which is the bending moment formula.

22. Potential Energy of Bending.—If W_1 is the elastic potential energy of bending of a rod per unit length,

$$W_1 = \frac{1}{2}EI\left(\frac{d^2y}{dx^2}\right)^2.$$

In general,

$$\text{Work} = W = \int fds$$

where f = force and s = distance measured in the direction of the force. For work of bending, $W = \int Ld\theta$, where L = torque = bending moment M, and θ = angle through which the rod is bent, measured in the direction of the torque.

Thus in bending unit length of the rod through the angle θ,

$$W_1 = \int_0^{\theta_1} Md\theta = \int_0^{\theta_1} EI\theta d\theta = \frac{EI\theta_1{}^2}{2}$$

$$= \frac{1}{2}\frac{EI}{\rho^2} = \frac{1}{2}EI\left(\frac{d^2y}{dx^2}\right)^2 \quad . \quad . \quad (24)$$

The substitutions used are evident by referring to the previous section.

This is a useful formula in certain types of bending vibrations.

CHAPTER IV

SIMPLE TORSIONAL VIBRATION; MOMENT OF INERTIA; VIBRATION WITH ONE DEGREE OF FREEDOM AND DAMPING

23. Simple Torsional Vibration.—The formula for a simple torsional vibration is derived in a manner exactly similar to that for a simple linear vibration given in § 10 and is entirely analogous to it.

For linear vibratory motion as in § 10,

$$\text{Force} = \frac{W}{g}a = \frac{W}{g}\frac{d^2y}{dt^2} = -ky \quad . \quad . \quad . \quad (25)$$

For the corresponding angular motion, such as that for the disk supported on a circular rod, performing a torsional vibration as shown in Fig. 14,

$$\text{Torque} = L = \frac{I}{g}\alpha = \frac{I}{g}\frac{d^2\theta}{dt^2} = -L_1\theta \quad . \quad . \quad (26)$$

as in § 10.

In the first case, a = linear acceleration and k = linear elastic constant.

In the second case, L = torque, α = angular acceleration, and L_1 = angular elastic constant = torque per radian of twist.

Thus, for the angular oscillation, in correspondence to the linear of § 10,

$$\text{Frequency} = f = \frac{1}{2\pi}\sqrt{\frac{L_1 g}{I}} \quad . \quad . \quad . \quad . \quad (27)$$

The quantity I corresponds to W of the first case and is recognized as the moment of inertia of the oscillating

mass. In Fig. 14 it is the moment of inertia of the disk about its central axis. The moment of inertia of a body about a given axis may be thought of as its angular mass about that axis.

24. Torsional Elastic Constant of Rod.—The torsional elastic constant of the circular rod of Fig. 14 is found as follows:

Since L_1 is defined as the torque per radian of twist for the whole rod of length l, it is evident that the section of

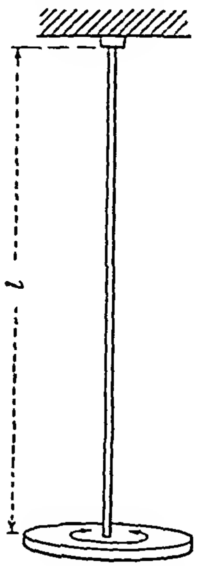


FIG. 14.—Torsional Vibration.

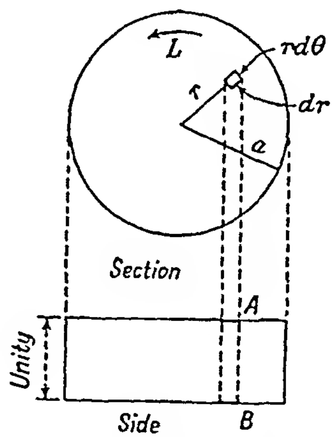


FIG. 15.—Section of Twisted Rod.

unit height shown in Fig. 15 is twisted through $1/l$ radians when L_1 is applied.

The elementary area $rd\theta dr$ at A in the side view of the unit section (Fig. 15) is sheared across the corresponding area at B by an amount determined by the product of the angle $1/l$ and the distance r of this area from the neutral point at the cylinder axis. Since this is by definition the shearing strain ϵ_s (see § 15),

$$\epsilon_s = \frac{r}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Since by § 15, the shearing stress $\tau = N\epsilon_s$, where N is the modulus of shear,

$$\tau = \frac{Nr}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where τ = the intensity of shear or the shear per unit area at the point where the elementary area was taken. Therefore the actual shearing force on this elementary area is

$$\tau r d\theta dr = \frac{Nr^2 d\theta dr}{l} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The torque dL_1 produced by this force is equal to the force times its distance r from the axis, or

$$dL_1 = \frac{Nr^3 d\theta dr}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Therefore,

$$L_1 = \int_0^{2\pi} \int_0^{a} \frac{Nr^3 d\theta dr}{l} = \frac{\pi N a^4}{2l} \quad . \quad . \quad . \quad (32)$$

where a = radius of rod.

This is the desired value of L_1, the elastic constant of the rod.

Substituting (32) in (27) it is found that the frequency of the system of Fig. 14 equals

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi N a^4 g}{2lI}} = \frac{a^2}{2\pi}\sqrt{\frac{\pi N g}{2lI}} \quad . \quad . \quad . \quad (33)$$

25. Moment of Inertia.—The physical meaning of the moment of inertia or angular mass of a rigid body may be well shown as follows.

Consider a rigid, weightless rod pivoted at O, Fig. 16, with a small weight w on its end. A torque L about O produces a force f on w such that

$$f = \frac{w}{g}\frac{d^2y}{dt^2}$$

where y is measured along the path of w.

But

$$L = fr = \frac{wr}{g}\frac{d^2y}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Also if θ = angle swept out by r when w moves a distance y, $y = r\theta$, so that

$$\frac{d^2y}{dt^2} = r\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Substituting (35) in (34)

$$L = \frac{wr^2}{g}\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Comparing (36) with (26) it is seen that the moment of inertia I of the simple system of Fig. 16 about $O = wr.^2$

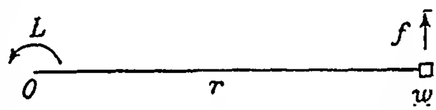


FIG. 16.—Simple Rotating System.

It is evident that an extended rigid body may be regarded as made up of a large number of small weights w, each at its own distance r from the axis of rotation so that for the whole body I = sum of all these quantities.

Expressed mathematically,

$$I = \sum wr^2 = W\bar{r}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

W is the total weight of the whole body and $\bar{r}$ a particular value of r such that, if W were concentrated at a point, $\bar{r}$ would be the distance from W to the axis of rotation. The distance $\bar{r}$ is called the *radius of gyration*, and is seen to be the root mean square value of r determined by equation (37), when the body is of constant density.

In the exact process of evaluating I the summation indicated in equation (37) is performed by an integration, the component weights w being taken as infinitesimally small. The expression for I in this case is written

$$I = \int\int\int r^2 \rho dV = Wr^2 \quad . \quad . \quad . \quad (38)$$

where ρ is the density of the material and dV an element of volume.

26. Routh's Rule.—It is seen that when the radius of gyration $\bar{r}$ of the body about its axis of rotation is once known, I is easily found by multiplying $\bar{r}^2$ by W, the weight of the body.

A very useful rule for finding $\bar{r}^2$ about an axis of symmetry of an isotropic body, without an integration, is given by Routh, being known as *Routh's rule.* It is stated as follows:

$$\bar{r}^2 = \frac{\text{sum of the squares of the perpendicular semi axes}}{3,\ 4,\ \text{or}\ 5}$$

the denominator to be 3, 4, or 5 according as the body is a rectangle, ellipse (including circle), or ellipsoid (including sphere).

For instance, for a rectangle with axis through O perpendicular to the paper, Fig. 17,

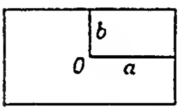


FIG. 17.—Rectangle

$$\bar{r}^2 = \frac{a^2 + b^2}{3} \quad \ldots \ldots \ldots \quad (39)$$

If the axis coincides with b,

$$\bar{r}^2 = \frac{a^2 + 0}{3} = \frac{a^2}{3} \quad \ldots \ldots \quad (40)$$

where the thickness of the rectangle squared can be neglected in comparison with a^2.

For a cylinder of radius r about its axis,

$$\bar{r}^2 = \frac{r^2 + r^2}{4} = \frac{r^2}{2} \quad \ldots \ldots \quad (41)$$

27. One Degree of Freedom with Damping.—Consider a system of one degree of freedom like that shown in Fig. 18, where, in addition to a spring of elastic constant k, a dash pot is introduced in parallel with the spring. The friction thus produced is assumed to be directly proportional to the velocity of motion. This law of friction is

taken because, besides being characteristic of dash pot action, this law permits of simple analytic treatment.

If b signifies force per unit velocity, the force produced by the velocity $\frac{dy}{dt}$ is $b\frac{dy}{dt}$. The force produced by the spring is ky.

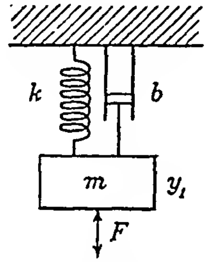


FIG. 18.—One Degree of Freedom with Damping.

These two external forces which act on the weight are called the impressed forces. It is the impressed forces which cause the weight to accelerate in the direction in which they act. Both of the impressed forces are negative in this case since when the velocity and displacement are taken positive they both act toward the origin or point of zero displacement, giving an acceleration towards the origin. Expressed mathematically,

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ky$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad . \quad . \quad . \quad . \quad (42)$$

The symbol m = mass is used in place of W/g to simplify the notation.

A solution of this equation is

$$y = Ae^{-\frac{bt}{2m}} \sin(\omega_0 t + \theta_1) \quad . \quad . \quad . \quad . \quad (43)$$

where $\omega_0 = \dfrac{\sqrt{4mk - b^2}}{2m}$ and where A and θ_1 are arbitrary constants determined, respectively by the initial amplitude and phase angle when $t = 0$.

Note that the vibration is harmonic and gradually dies out with time. If $b = 0$, expression (43) reduces to the form of equation (4), the vibration continuing indefinitely, and ω_0 reduces to $\omega_c = \sqrt{k/m}$.

28. Sustained Vibration with One Degree of Freedom and Damping.—For the case of a sustained vibration with one degree of freedom with damping, consider the case of Fig. 18 with a periodic force $F \cos \omega t$ applied to the mass in an up-and-down direction. The stimulating force is an additional impressed force and has a positive sign, because it gives m a positive acceleration component. With this additional force, equation (42) becomes

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F \cos \omega t \quad . \quad . \quad . \quad (44)$$

The solution of this equation is

$$y = Ae^{-\frac{bt}{2m}} \sin (\omega_0 t + \theta_1) + \frac{F \cos (\omega t - \beta)}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \quad . \quad (45)$$

where

$$\tan \beta = \frac{b\omega}{k - m\omega^2} \quad \text{and} \quad \omega_0 = \frac{\sqrt{4mk - b^2}}{2m}$$

and where A and θ_1 are constants depending on initial amplitude and phase.

The vibration is seen to be made up of two components, one on top of the other. The first is the free vibration component, whose amplitude and phase depend on how the vibration was started, and which rapidly dies out with time. The second is the forced or sustained vibration component which always has the same amplitude regardless of the initial conditions.

As an example, consider the special case where

$$y = 0 \quad \text{and} \quad \frac{dy}{dt} = 0 \quad \text{when} \quad t = 0 \quad \text{and} \quad \omega = \omega_c = \sqrt{\frac{k}{m}}$$

From these conditions,

$$\theta_1 = 0, \quad \beta = \frac{\pi}{2} \quad \text{and} \quad A = -\frac{F}{b\omega_0}$$

Substituting these quantities in (45),

$$y = -\frac{F}{b\omega_0} e^{-\frac{bt}{2m}} \sin \omega_0 t + \frac{F \sin \omega_c t}{b\omega_c} \quad . \quad . \quad (46)$$

This represents a vibration which gradually builds up to the value

$$y = \frac{F \sin \omega_c t}{b\omega_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

which is reached when the transient term drops out. Since b is small in ordinary cases, ω_c and ω_0 are very nearly equal to each other so that by the time the phase of the two components becomes opposite the amplitude of the transient has become almost zero.

CHAPTER V

SOLUTION OF VIBRATION PROBLEMS BY THE USE OF COMPLEX QUANTITIES WITH APPLICATIONS; VIBRATION OF COMPOUND SYSTEM

29. Use of Complex Notation.—The use of complex notation in solving for the steady state is well illustrated by the example of the previous section. Instead of representing the exciting force by $F \cos \omega t$ it is taken as $F\epsilon^{j\omega t}$ where $j = \sqrt{-1}$. Since imaginary quantities are of an entirely different nature from real, the usual convention in graphical representation is to plot them along the vertical axis and the real quantities along the horizontal axis so they are independent of each other. In Fig. 19, the quantity $F \cos \omega t$ is given by the projection of the revolving vector F on the real or horizontal axis, whereas the quantity $F\epsilon^{j\omega t} = F(\cos \omega t + j \sin \omega t)$ includes the imaginary component as well as the real component of F, and thus represents the complete revolving vector. It is easily shown that

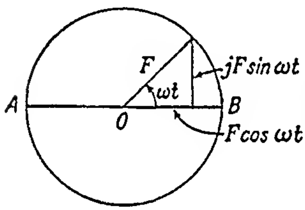


FIG. 19.—Real and Imaginary Components.

$$\left.\begin{aligned} \epsilon^{j\omega t} &= \cos \omega t + j \sin \omega t \\ \text{and} \qquad \epsilon^{-j\omega t} &= \cos \omega t - j \sin \omega t \end{aligned}\right\} \quad . \quad . \quad . \quad (48)$$

by expanding the separate terms into their corresponding series, and comparing the resulting right- and left-hand members. If ωt is measured in a counter-clockwise direction, $\epsilon^{j\omega t}$ represents a unit vector revolving in the counter-clockwise direction and $\epsilon^{-j\omega t}$ a unit vector revolving in the clockwise direction.

Equation (44) may thus be written as

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = Fe^{j\omega t} \quad . \quad . \quad . \quad . \quad (49)$$

regarding each term as a vector. Neglecting the transient state, the steady state of vibration may be assumed to be $y = Ae^{j\omega t}$, from which

$$\frac{dy}{dt} = j\omega y \quad \text{and} \quad \frac{d^2y}{dt^2} = -\,\omega^2 y$$

which shows the phase relations of the individual terms on the basis of the solution assumed.

Substituting these quantities in (49), there is obtained

$$(-m\omega^2 + jb\omega + k)\,y = Fe^{j\omega t} \quad . \quad . \quad . \quad (50)$$

Thus

$$y = \frac{F}{Z'}e^{j\omega t} \quad \text{where} \quad Z' = -\,m\omega^2 + k + jb\omega$$

The solution assumed is correct provided $A = \frac{F}{Z'}$, where Z' is a complex quantity, so that A must be one also.

The complex quantity Z' will be called the *mechanical impedance* of the vibrating system after A. G. Webster (reference 3). The mechanical impedance is that quantity by which F must be divided to give the vibration amplitude. The complex character of the amplitude factor merely means that it differs in phase from the exciting force F.

Let us find the phase angle. This complex quantity A is of the form

$$\frac{F}{a + jb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where a is the real part and b the imaginary part of the mechanical impedance. Multiplying the numerator and denominator each by $a - jb$, there is obtained

$$F\cdot\frac{a - jb}{a^2 + b^2} = \frac{F}{\sqrt{a^2 + b^2}}\left(\frac{a}{\sqrt{a^2 + b^2}} - j\frac{b}{\sqrt{a^2 + b^2}}\right)$$

$$= \frac{F}{\sqrt{a^2 + b^2}} (\cos \beta - j \sin \beta) = \frac{Fe^{-j\beta}}{\sqrt{a^2 + b^2}} \quad . \quad (52)$$

where $\tan \beta = \frac{b}{a}$ is $\frac{b\omega}{k - m\omega^2}$ for the problem just considered. Thus (51) represents a vector of magnitude $\frac{F}{\sqrt{a^2 + b^2}}$ which lags behind the vector F by an angle β (see Fig. 19). Note that the solution of (44) given by equation (45) shows exactly this result for the steady state term as obtained without the use of complex quantities.

30. Remarks on the Complex Quantity Method.—The complex quantity method, in mathematical language, is a shorthand process for solving for the particular integral of the linear differential equations of vibratory motion where restoring forces are directly proportional to displacements. The motions of any vibrating system with lumped masses, stiffness factors, and friction coefficients can be completely expressed by such a set of simultaneous differential equations, the preceding section containing an example for one degree of freedom. The same may be said of alternating-current electrical circuits with lumped inductances, capacities, and resistances, analogous respectively to masses, elastic coefficients (reciprocal), and friction constants of mechanical vibrating systems. The use of complex quantities in alternating-current electrical circuits is thus completely analogous to the process presented in the preceding section. The alternating electric current amplitudes correspond to alternating velocity amplitudes of the masses of the mechanical system, and alternating charge displacement amplitudes correspond to mass displacement amplitudes of the mechanical system. The mechanical impedance is seen to differ from electrical impedance by the factor $j\omega$, because in mechanical systems

the impedance is so defined that force times impedance gives amplitude, whereas in electrical systems force times impedance gives current, which is the analogue of velocity. Referring to § 29 it is seen that sinusoidal amplitude and velocity differ by the factor $j\omega$, the latter being the derivative of the amplitude, where j takes account of the 90° phase displacement between the two.

Note that this process applies only where the steady state of vibration is required. When a mechanical system is acted upon by an alternating force of steady amplitude, in ordinary cases transient vibrations drop out in a few moments and are of no particular importance. The same is true for alternating-current circuits. Thus a process which neglects transients is sufficient.

In solving mechanical systems it is best to proceed directly without the use of equivalent electric circuits, which are a superfluity, though often helpful to one versed in electrical theory.

The impedance must be a complex quantity in general because the periodic force and the resulting motions are vector quantities having different phase relations. The equations which are solved to obtain amplitudes are therefore vector equations, the impedance being a complex multiplier which makes the two sides of each equation equal in both phase and magnitude.

31. Rules for Solving Problems.—Draw a sketch of the mechanical system, as shown, for example, in Fig. 22, where the dash-pot indicates damping. Specify the amplitude of motion of each junction of elements by y_1, y_2, y_3, etc. A mass has only one point of application while springs and dash-pots have two. The intersections of all springs and dash-pots attached to a mass are identical. A spring and a dash-pot may be connected without any mass at the junction. Note from § 29 that the mechanical impedance of a single mass m is $-m\omega^2$, that of a single elastic connection $= k$, and that of a single damping factor (friction proportional to velocity) $= jb\omega$. Set up a set of

force equations, one for each junction of elements according to the following rules:

Rule 1. Multiply the amplitude y of the junction by every impedance element which is connected to it.

Rule 2. Subtract from this quantity the product of the amplitude of the farther terminal of each spring or dash-pot into their respective impedances.

Rule 3. Set this quantity equal to the force applied at the intersection if it is present. If more than one force is applied to the system, account must be taken of their phase relation.

A set of linear equations in y_1, y_2, etc., will be obtained whose number equals the number of individual junctions of the system, which can be readily solved for amplitudes by determinants.

Rule 4. To find the phase angle of the displacement with respect to the exciting force, if required, put the expression for that displacement in the form $y = F/(A + jB)$ which always can be done. Then, as outlined in § 29, y has the magnitude $F/\sqrt{A^2 + B^2}$ and lags behind F by the angle β where $\tan \beta = B/A$.

Any problem of this type can be solved by this process, and most practical problems with one degree of freedom will be found to solve very easily (reference 4).

Note that this analysis applies to motions which take place only in one independent coordinate (y in this case). Motion in other coordinates is treated separately when required.

32. Illustrations of Set-up of Equations.

1. Periodic force $F \cos \omega t$ acting on a mass m, Fig. 20. Let amplitude of $m = y_1$. Then

$$-y_1 m\omega^2 = F, \quad y_1 = \frac{-F}{m\omega^2} \quad . \quad . \quad . \quad (53)$$

This equation gives the amplitude y_1 with a minus sign which means that force and displacement are in phase opposition. If a weight held in the hand is rapidly oscillated to the right and left this phase opposition is at once perceived in that the maximum *right-hand force* is exerted when the weight is in the extreme *left position*, and vice versa.

2. Periodic force $F \cos \omega t$ acts on spring, Fig. 21.
In this case

$$y_1 k = F; \quad y_1 = \frac{F}{k}$$

Here the force and displacement are in phase, as a little consideration will show must be true.

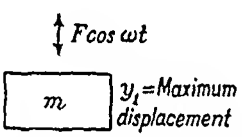


FIG. 20.—Vibration of Single Mass.

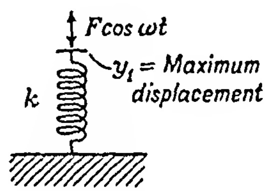


FIG. 21.—Vibration of Single Spring.

3. Take the case of Fig. 18, where a periodic force F acts upon a mass m supported by a spring k, with a dash pot of friction coefficient b as shown. Since there is one moving junction point, one equation is sufficient. By rule 1, we have

$$y_1(k - m\omega^2 + jb\omega) = F$$

No further terms are required because the lower ends of k and b are fixed. (See rule 2.)

By rule 3 the left-hand member is set equal to F. Solving for y_1

$$y_1 = \frac{F}{k - m\omega^2 + jb\omega} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

By rule 4

$$y_1 = \frac{F \overline{\diagdown \beta}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \text{ where } \tan \beta = \frac{b}{k - m\omega^2}$$

and where $\overline{\diagdown\beta}$ means the phase angle β is measured backwards against time, that is, y_1 lags with respect to F. This problem is seen to be the same as the one treated in § 29.

4. Take the case of the previous problem except that k and b are in series as shown in Fig. 22.

In this case there are two moving junction points, one at m whose amplitude of motion is specified by y_1 and one between the spring and dash-pot whose motion is specified by y_2.

Using rules 1, 2 and 3, the following pair of simultaneous equations is obtained:

$$y_1(jb\omega - m\omega^2) - y_2 jb\omega = F$$

$$y_2(k + jb\omega) - y_1 jb\omega = 0.$$

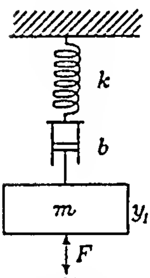


FIG. 22.—Spring and Dash-pot in Series.

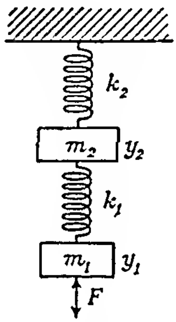


FIG. 23.—Compound System.

From the solution of these equations the displacement amplitudes y_1 and y_2 can be found, and by applying rule 4 their phase angle with respect to F is found.

33. Compound System.—The amplitudes of vibration for the steady state of the compound system of two degrees of freedom shown in Fig. 23 are easily found by the use of complex quantities.

The general procedure in a case of this sort is as follows.

Use a separate symbol for the amplitude of motion of each mass. In this case y_1 is the amplitude of m_1 and y_2 that of m_2.

Set up a separate equation for the forces on each mass. In this case, by rules 1, 2, and 3, § 31

$$\left.\begin{aligned} y_1(-m_1\omega^2 + k_1) - y_2k_1 &= F \\ y_2(-m_2\omega^2 + k_1 + k_2) - y_1k_1 &= 0 \end{aligned}\right\} \quad . \quad . \quad (55)$$

In this problem no damping is present, so that the symbol j does not enter.

By rule 1, the amplitude of each mass is multiplied by each impedance which is connected with it. For the mass m_1 this gives $y_1(-m_1\omega^2 + k_1)$. Then from this quantity is subtracted the product of each of these impedance components into the amplitude of their other terminals. For m_1 there is only one such term, $-y_2k_1$. Set the sum of these terms equal to the exciting force F which acts on m_1. This accounts for the first equation. The second is obtained by the same process. Three impedances are seen to act at m_2. Only the spring of elastic constant k_1 has its farther terminal move a finite distance, which terminal moves through the amplitude y_1 so that the term y_1k_1 is subtracted in this case.

These rules may easily be established by setting up simultaneous differential equations in the usual way and making substitutions as in equation (50) and rearranging terms.

The amplitudes are most easily found by solving the simultaneous equations by the use of determinants. For this case, first arrange the equations in symmetrical form, using a_1, b_1, a_2, and b_2 as coefficients.

$$\left.\begin{aligned} a_1y_1 + a_2y_2 &= F \\ b_1y_1 + b_2y_2 &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (56)$$

Then solve for y_1 and y_2 in the usual way.

$$\left.\begin{aligned} y_1 &= \frac{\begin{vmatrix} F & a_2 \\ 0 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{Fb_2}{a_1b_2 - a_2b_1} \\ y_2 &= \frac{\begin{vmatrix} a_1 & F \\ b_1 & 0 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{-Fb_1}{a_1b_2 - a_2b_1} \end{aligned}\right] \quad . \; . \; . \; . \; . \; . \quad (57)$$

Substituting,

$$\left.\begin{aligned} y_1 &= \frac{F(-m_2\omega^2 + k_1 + k_2)}{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_1 + k_2) - k_1^2} \\ y_2 &= \frac{Fk_1}{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_1 + k_2) - k_1^2} \end{aligned}\right\} \qquad (58)$$

The resonant frequencies of this system for which both y_1 and y_2 become infinite are found by setting the denominator equal to zero. This gives

$$f = \frac{1}{2\pi\sqrt{2}}\sqrt{\frac{k_1}{m_2}+\frac{k_2}{m_2}+\frac{k_1}{m_1}\pm\sqrt{\left(\frac{k_1}{m_2}+\frac{k_2}{m_2}+\frac{k_1}{m_1}\right)^2-4\frac{k_1k_2}{m_1m_2}}} \qquad (59)$$

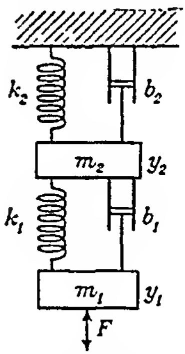


FIG. 24.—Compound System with Damping.

The system thus has two resonant frequencies, for the lower of which the two weights vibrate in phase and for the higher of which they vibrate in phase opposition. This is seen to be the same system as that discussed in § 12.

34. Compound System with Damping.—This system is shown in Fig. 24. Using the general rules of § 31, the force equations corresponding to (55) become

$$\left.\begin{aligned} y_1(-m_1\omega^2+k_1+jb_1\omega)-y_2(k_1+jb_1\omega) &= F \\ y_2(-m_2\omega^2+k_1+k_2+jb_1\omega+jb_2\omega)-y_1(k_1+jb_1\omega) &= 0 \end{aligned}\right\} \qquad (60)$$

The solution of these equations is carried out in exactly the same way as outlined in the previous section.

With a little practice the equations can be readily set up for any system of this type, however complex. The actual expressions for amplitude may be very complicated indeed. Nevertheless this is the simplest and most direct method known for handling such problems.

CHAPTER VI

LONGITUDINAL AND FLEXURAL VIBRATIONS OF RODS

35. Longitudinal Vibration of Rods.—Before taking up flexural vibrations of rods which are important in vibration work the interesting case of longitudinal vibrations of rods will be treated, which is somewhat simpler than the former.

If a tension is applied in the X direction on the piece of rod shown in Fig. 25 fixed at the origin the section AB will be displaced a small distance to the position CD, owing to the elastic strain. This displacement is called the total strain at AB and will be designated by the symbol u. It is important to appreciate that this differs from the strain per unit length ϵ which determines the stress intensity. It is evident from definition that

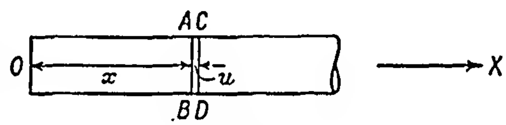


FIG. 25.—Longitudinal Strain in Rod.

$$\text{Strain} = \epsilon = \frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

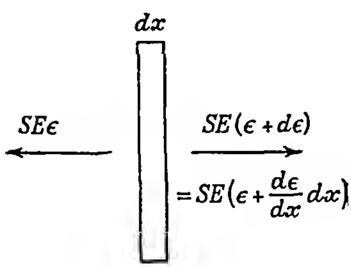


FIG. 26.—Section of Rod.

Let us find what forces act on the element or section of the rod of thickness dx shown in Fig. 26 enlarged. If S = cross-sectional area of the section and the strain ϵ increases in the positive direction of X the force on each side of the section due to the elastic strain is as shown in the figure. The strain ϵ is assumed positive so that the stress is a

tension. The resultant external force acting on this section is

$$SE\left(\epsilon + \frac{\partial \epsilon}{\partial x} dx\right) - SE\epsilon = SE\frac{\partial \epsilon}{\partial x} dx \quad . \quad . \quad (62)$$

If ρ = mass per unit volume of the rod material, the mass of the slice of Fig. 26 $= \rho S dx$. Therefore

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = SE \frac{\partial \epsilon}{\partial x} dx = SE \frac{\partial^2 u}{\partial x^2} dx$$

This is the mathematical statement that the mass of the slice times its acceleration is equal to the resultant external force on it.

Rearranging and cancelling common factors,

$$\rho \frac{\partial^2 u}{\partial t^2} - E \frac{\partial^2 u}{\partial x^2} = 0. \quad . \quad . \quad . \quad . \quad (63)$$

Although here derived for a bar fixed at one end, the general solution of this differential equation gives all possible kinds of longitudinal elastic waves in such a rod, regardless of end conditions.

This vibration system differs from any thus far considered in that, instead of separate masses joined by elastic connections, a continuous elastic medium of uniformly distributed mass is involved. The vibration characteristics of such a medium are the same as though it consisted of a very large number of very small separate masses all connected by elastic bonds of equal strength. The continuous medium is the limiting case where the number of the component masses is indefinitely great and their size indefinitely small. A system with an infinite number of elastically connected masses should have an infinite number of degrees of freedom. It is very interesting that this is just what the solution of equation (63) shows.

36. Solution of Equation for Longitudinal Vibrations.—The form of solution of equation (63) which satisfies the requirements of this problem is

$$u = A_1 e^{j\omega t + j\beta x} + A_2 e^{j\omega t - j\beta x} \quad . \quad . \quad . \quad (64)$$

where A_1 and A_2 and β depend upon the end conditions of the rod, and ω, the frequency constant $= 2\pi f$, depends upon its physical properties. This solution is given directly without going into the method of obtaining it, which can be found in books devoted to differential equations.

For a free-free bar (both ends free), since the stress $E\epsilon$ must be zero at the ends,

$$\epsilon = \frac{du}{dx} = 0 \text{ when } x = 0 \text{ and when } x = l \quad . \quad (65)$$

where l = length of the rod.

$$\frac{\partial u}{\partial x} = j\beta A_1 e^{j\omega t + j\beta x} - j\beta A_2 e^{j\omega t - j\beta x} = 0 \text{ when } x = 0$$

Therefore, $A_1 = A_2 = A/2$, say.

Then from (64),

$$u = \frac{A}{2} e^{j\omega t}(e^{j\beta x} + e^{-j\beta x}) \quad . \quad . \quad . \quad . \quad (66)$$

From equations (48)

$$u = Ae^{j\omega t} \cos \beta x = A(\cos \omega t \cos \beta x + j \sin \omega t \cos \beta x). \quad (67)$$

Equation (67) gives the amplitude of vibration u as a vector revolving at a constant angular velocity ω whose length varies along the rod according to the law $\cos \beta x$. (See § 29.)

Since the actual vibration amplitude is the projection of this vector on the axis of real quantities, the j component of (67) can be dropped. Its use is of value merely in simplifying the process of solution of the differential equation. In specifying the amplitude u it has no physical meaning. When the j component is dropped, (67) becomes

$$u = A \cos \omega t \cos \beta x \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

When (68) is plotted it is found to represent a series of standing waves along the rod, as shown in Fig. 27, where the amplitude u is plotted along the Y axis. These

waves rise and fall between the maximum and minimum positions shown in the figure with fixed nodal points.

If u_0 = amplitude when $t = 0$ at $x = 0$, from (68) $A = u_0$, so that

$$u = u_0 \cos \omega t \cos \beta x \quad . \quad . \quad . \quad . \quad (69)$$

Also

$$\frac{du}{dx} = -\ u_0\beta \cos \omega t \sin \beta x$$

which, from (65) when $x = l$,

$$= -\ u_0\beta \cos \omega t \sin \beta l = 0$$

This requires that $\sin \beta l = 0$, since $\cos \omega t$ is not always zero. Therefore

$$\beta l = n\pi; \quad \beta = \frac{n\pi}{l} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where n has any integer value.

Thus from (69) when $x = l$, $u = u_0$, as indicated in Fig. 27.

Since β may have any one of an infinite number of values the rod may vibrate in any one of an infinite number of modes of which Fig. 27 represents the particular case where $n = 4$. Of course, the general solution (69) is the sum of all harmonics corresponding to all values of n. These may, as brought out in § 12, be superimposed upon each other in any combination without their individual frequencies being affected.

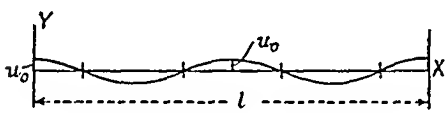


Fig. 27.—Vibration in Four Nodes.

From Fig. 27 a complete wave length, designated by λ, is equal to twice the distance between nodes. It equals the value of x corresponding to one cycle of $\cos \beta x$, that is

$$\text{when } \beta x = 2\pi; \quad x = \lambda = \frac{2\pi}{\beta} = \frac{2l}{n} \quad . \quad . \quad (71)$$

since from (70), $\beta = \dfrac{n\pi}{l}$.

Thus for Fig. 27 where $n = 4$, $\lambda = l/2$.

The only constant left to evaluate is the frequency $f = \omega/2\pi$. This is evaluated by substituting the solution (69) in the original differential equation (63), which it must satisfy. This gives the relation

$$-\rho u_0 \omega^2 \cos \omega t \cos \beta x + E u_0 \beta^2 \cos \omega t \cos \beta x = 0$$

from which

$$\omega = \beta \sqrt{\frac{E}{\rho}} = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$

Therefore

$$f = \frac{n}{2l} \sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

It is seen from (72) that each harmonic has a different frequency, proportional to the number of the harmonic or to the number of fixed nodes. For the case of Fig. 27 where $n = 4$,

$$f = \frac{2}{l} \sqrt{\frac{E}{\rho}}$$

Now, for a progressive train of waves, evidently frequency equals velocity of motion of the wave train divided by wave length, or expressed mathematically

$$f = \frac{V}{\lambda}$$

where V = velocity of the wave train.

From (71) and (72)

$$V = \sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

This equation gives the velocity of transmission of a wave train of longitudinal vibrations along the rod. It is seen to depend only upon the elastic modulus E and the density ρ of the material of the rod. It is really the velocity of transmission of sound along the rod, which for steel is equal to about 17,000 feet per second, or about 16 times that of sound in air.

37. Travelling and Standing Waves.—It will be here shown that a train of standing waves such as that obtained in the solution of the problem of the previous section results from the superimposition of two trains of travelling waves of equal wave length moving in opposite directions (reflecting back and forth along the rod).

Let Fig. 28 represent part of a moving wave train here shown when one of the travelling nodes is passing through the origin.

If the wave were fixed in this position its equation would evidently be

$$y = \frac{y_0}{2} \sin \frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (74)$$

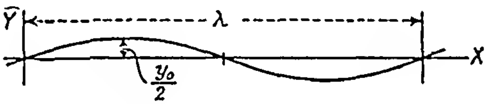


FIG. 28.—Travelling Wave.

It is seen that when the distance x is a multiple of the wave length λ, the wave angle is a multiple of 2π, as it should be.

In order that this wave train shall move to the right in the positive x direction the wave angle corresponding to any point on the x axis such as the origin must continuously *decrease* so that more and more negative portions of the wave progressively pass it. If this wave velocity is V, this decrease of wave angle per second must be Vt/λ, so that the equation for a wave moving to the right with the velocity V becomes

$$y = \frac{y_0}{2} \sin \frac{2\pi}{\lambda} (x - Vt) \quad . \quad . \quad . \quad . \quad . \quad (75)$$

In like manner, if the wave train is moving to the left, the wave angle corresponding to each point x must progressively *increase* so that the equation for this wave train is

$$y = \frac{y_0}{2} \sin \frac{2\pi}{\lambda}(x + Vt) \quad . \quad . \quad . \quad . \quad . \quad (76)$$

To find the type of wave produced by superimposing these two wave trains, add (75) and (76) after putting in the form for the difference and sum of two angles. The resulting equation is

$$y = y_0 \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi Vt}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

This equation is seen to be identical in form to (69), which has already been shown to be the equation of a standing wave train. Since, from the previous section, $\lambda = 2\pi/\beta = V/f$, making these substitutions, (77) becomes

$$y = y_0 \sin \beta x \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

It is often very useful in vibration work to separate such a standing vibrating wave train into its two oppositely moving wave trains. The analysis just given furnishes the basis for this.

Note also that a continuous and complete reflection of these wave trains from the ends of the bar is necessary for this agreement.

In the previous section the velocity of wave motion $V = \sqrt{E}/\rho$ was obtained from a standing wave train. When it is appreciated that this velocity is the actual velocity of the two oppositely moving wave trains of which the standing wave train is composed, this velocity assumes greater physical reality.

The expression for the velocity of sound in air is the same as (73), where E is the elasticity of the air.

The isothermal elasticity of air equals the pressure p.

Since in sound waves the compressions and rarefactions are relatively instantaneous, the adiabatic elasticity must be used $= \gamma p$ where $\gamma = c_p/c_v$, the ratio of specific heat of constant pressure to specific heat of constant volume. For air, formula (73) becomes

$$V = \sqrt{\frac{\gamma p}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

If mass per unit volume w/g is substituted in place of ρ, equation (79) becomes

$$V = \sqrt{\frac{\gamma p g}{w}} \qquad (80)$$

Another point of interest is that when a wave of compression travels along a rod of rigid material like steel it is attended by a wave of lateral expansion of small magnitude because of Poisson's ratio. This means that waves of sound in an extensive rigid medium travel at a slightly higher velocity than that given by (73) because in such a case no lateral expansion is possible, thus giving a slightly higher effective value of E. The analysis of this case can be found in works on elasticity (reference 5).

38. Flexural Vibrations of Rods.—Let Fig. 29 represent part of a rod or bar subject to transverse forces in the Y

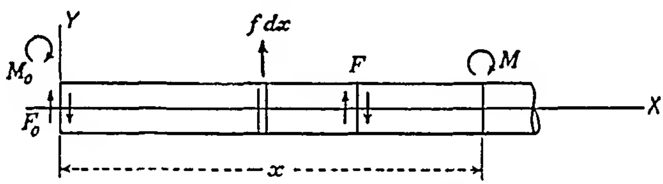


FIG. 29.—Forces in Flexural Vibration.

direction, the axis of which coincides with the X axis when it is undeflected.

Let f = transverse force per unit length, positive upwards; F the shearing force on any section, positive as shown; and M the bending moment on each section, positive when acting clockwise on a section as viewed in the positive X direction.

The shear on any section evidently equals the integral of fdx from the origin to that section plus the shear F_0 at the origin as seen from the figure. Expressed mathematically,

$$F = \int f dx + F_0 \qquad (81)$$

Also the change in bending moment M per unit length at

any point equals the shear at that point. This can be seen to be true because unit length has unit moment arm. Expressed mathematically,

$$\frac{dM}{dx} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

From (81), $dF/dx = f$, so that

$$\frac{d^2M}{dx^2} = f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Now since, from (23),

$$M = EI\frac{d^2y}{dx^2}$$

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = f \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Thus the fourth derivative of the deflection curve at any point is equal to the load per unit length at that point. The relation (84) will be made use of in analyzing flexural vibrations of rods.

Assume the rod to be performing sinusoidal or harmonic flexural vibrations.

Thus the motion of the rod at some point x is equal to $y = u \cos \omega t$, where $u =$ the maximum value of y or the amplitude of the motion. If $w =$ weight per unit length of the rod the weight of an element of length dx equals wdx. Its acceleration due to the sinusoidal motion is

$$\frac{d^2y}{dt^2} = -\, u\omega^2 \cos \omega t$$

The force producing this acceleration of the weight wdx is

$$\frac{wdx}{g}\frac{d^2y}{dt^2} = -\frac{wdx}{g}u\omega^2 \cos \omega t \quad . \quad . \quad . \quad (85)$$

If f' = the accelerating force per unit length at that point, when $y = u$

$$-\frac{wdx}{g}u\omega^2 = f'dx$$

or

$$-\frac{w}{g}u\omega^2 = f' \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Notice from Fig. 30 that f' acts downwards in accelerating the element w towards the X axis. The force f is caused

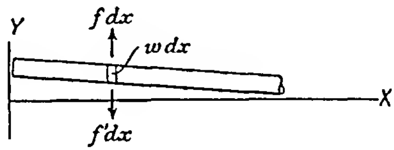


FIG. 30.—Action and Reaction of Element.

by the reaction of w against the rod structure and is thus equal and opposite to f'. Thus

$$\frac{w}{g}u\omega^2 = f \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

The differential equation from which the frequency constant ω can be found is obtained by substituting (87) in (84) and setting y equal to u, the maximum value of y for which (87) was derived. This gives

$$\frac{d^4u}{dx^4} - \frac{w\omega^2}{gEI}u = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Or

$$\frac{d^4u}{dx^2} - m^4u = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where

$$m = \left(\frac{w\omega^2}{gEI}\right)^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad (90)$$

39. Solution of Equation for Flexural Vibrations.—The general solution of (89) is well known to be

$$u = A\cosh mx + B\sinh mx + C\cos mx + D\sin mx. \quad (91)$$

Note that

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} \qquad (92)$$

For a bar fixed at one end and free at the other, called a fixed-free bar,

$$\left.\begin{array}{llll} \text{when } x = 0, & u = 0 & \text{and} & \dfrac{du}{dx} = 0 \\ \text{when } x = l, & \dfrac{d^2u}{dx^2} = 0 & \text{and} & \dfrac{d^3u}{dx^3} = 0 \end{array}\right\} \quad . \quad (93)$$

since at the free end of the rod the bending moment $M = 0$ and the shear $= dM/dx$ must also be zero.

Applying these four end conditions to (91), four separate equations of condition, as they are called, are obtained from which the four arbitrary constants A, B, C, and D can be eliminated, resulting in the final equation.

$$\cosh ml \cos ml + 1 = 0 \quad . \quad . \quad . \quad . \quad (94)$$

This equation is satisfied by an infinite number of values of ml, each of which corresponds to a particular mode of vibration. Since l is known, m can be found. The frequency of that mode can then be calculated from (90). The shape of the deflection curve at maximum amplitude can be found from (91) when the constants A, B, C, and D have been evaluated. (See § 88.)

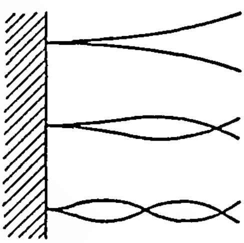


FIG. 31.—Modes of Vibration of Cantilever.

Figure 31 shows the first three modes of vibration with the corresponding values of the constant ml.

First mode, $ml = 1.875$
Second mode, $ml = 4.695$
Third mode, $ml = 7.856$

For higher modes, $ml = (2n - 1)(\pi/2)$, to a close approximation, n being the number of the harmonic.

For a supported-supported bar, the end conditions are

$$u = 0 \quad \text{and} \quad \frac{d^2u}{dx^2} = M = 0 \quad \text{when} \quad x = 0 \quad \text{or} \quad x = l \qquad (95)$$

These conditions applied as before yield the simple relation

$$\sin ml = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (96)$$

This gives $ml = n\pi$, where n = number of the mode.

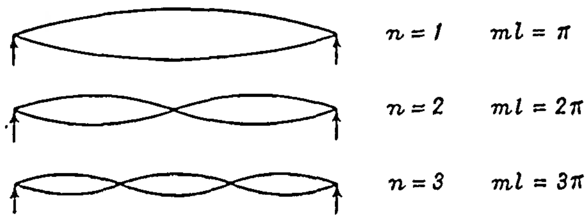


FIG. 32.—Modes of Vibration of Supported Rod.

Figure 32 shows the first three modes.

$$\begin{array}{ll} n = 1 & ml = \pi \\ n = 2 & ml = 2\pi \\ n = 3 & ml = 3\pi \end{array}$$

It will be shown later that this analysis holds for the critical speeds of a uniform shaft.

Note that in this analysis the cross-section of the bar is taken as uniform so that the second moment of cross-section I is a constant. If the rod is not uniform I will vary with x and the solution may be very complex or impossible.

For good treatments of the subject of vibrating bars see reference 6.

CHAPTER VII

THEOREMS ON THE ROTATION AND TRANSLATION OF RIGID BODIES AND APPLICATIONS

40. Theorems in Rigid Dynamics—Torque.—*A pure torque or couple acting on a rigid body can produce rotation about its center of gravity only, regardless of where the couple acts on the body.*

Let a couple act on the body shown in Fig. 33 about an axis parallel to the vector A. Assume that the Y axis coincides with the axis of rotation of the body. As the body is accelerated about the Y axis the element $dxdydz$ is acted upon by the force $df = (w/g)adxdydz$, where w = weight of unit volume and a = linear acceleration of the element in the direction Z.

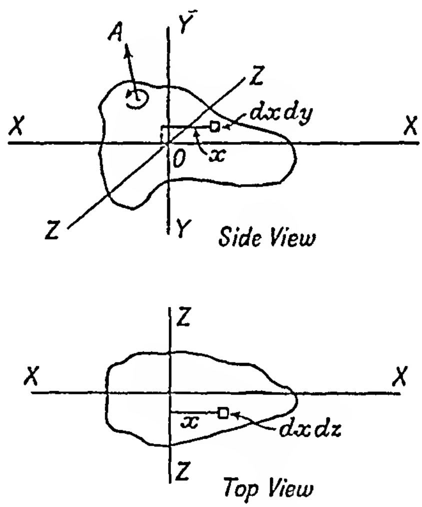


FIG. 33.—Torque Acting on Mass.

If α = angular acceleration, $a = \alpha x$. This is true whether the element be in the XY plane or not, as can be seen from the top view shown in the figure. Then

$$df = \frac{w\alpha}{g} xdxdydz \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Since for a pure torque no resultant translating force exists, the forward accelerating forces parallel to Z on

one side of the Y axis must equal the backward accelerating forces on the other side, so that

$$\frac{w\alpha}{g}\underset{(X\text{ positive})}{\int\int\int} xdxdydz = \frac{w\alpha}{g}\underset{(X\text{ negative})}{\int\int\int} xdxdydz \quad . \quad (98)$$

But this is the condition that the center of gravity lie in the YZ plane, namely, that the sum of every element multiplied by its perpendicular distance from one side of the plane equal the same sum for the other side.

By exactly similar reasoning the center of gravity may be shown to lie also in the XY plane, so that it must therefore lie on the Y axis. Since by hypothesis the Y axis is the axis of rotation, the center of gravity must lie on the axis of rotation, and vice versa.

41. Single Force.—*A single force acting on a rigid body can cause pure translation without rotation only when its line of action passes through the center of gravity of the body.*

Assume a system of coordinate axes to pass through the body as in Fig. 33, but in this case take the origin at the point through which the force must pass to cause pure translation and let the force act along the Z axis. Every element of the body is then subject to a linear acceleration a. In this case the reaction of each element causes a turning moment about the Y axis given by

$$dM = \frac{wa}{g} xdxdydz \quad . \quad . \quad . \quad . \quad . \quad (99)$$

For pure translation the sum of the moments produced by elements on each side of the Y axis must be equal and opposite so that

$$\frac{wa}{g}\underset{(X\text{ positive})}{\int\int\int} xdxdydz = \frac{wa}{g}\underset{(X\text{ negative})}{\int\int\int} xdxdydz \quad . \quad (100)$$

As in the previous section the center of gravity must lie in the YZ plane. In like manner it is also found to lie in the XZ plane, which two conditions require that it lie

on the Z axis. If now the force is applied along the Y axis a third condition is obtained which proves the origin to be the center of gravity of the body.

Since by hypothesis the origin was taken as the point through which the force must act to cause no translation, this point and the center of gravity are coincident, and vice versa.

Conversely a single force will cause a rigid body to rotate in all cases when its line of action does not pass through the center of gravity.

It is evident that if a force must act in line with the center of gravity of a body to cause linear acceleration of the body without rotation, the reaction to that force must also lie on a line through the center of gravity of the body.

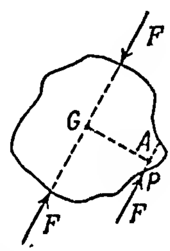


FIG. 34. — Single Force Acting on Mass.

Note from Fig. 34 that a single force F acting on a rigid body at P, say, along a line not passing through its center of gravity is equivalent to a couple consisting of a pair of forces F and moment arm AG together with a single translating force F passing through its center of gravity. If the body is placed in the figure so that the original force and G lie in the plane of the paper, the other two forces must also lie in that plane.

It follows from this that, no matter where a single force acts on a rigid body, it always causes a translation of the center of gravity of the body as though the force were acting at the center of gravity.

Rotation about the center of gravity is produced in addition because of a torque equal to the force times the distance of its line of action from the center of gravity.

A pure torque, however, can cause rotation only about the center of gravity with no additional translation, regardless of where the torque acts on the body. (See § 40.)

42. Further Remarks on the Motion of Rigid Bodies.— It is well known that a spinning body rotates about an

axis through its center of gravity. It is readily shown by a method similar to that used in § 40 that this is necessary if the radial centrifugal forces are to be in equilibrium.

It should be pointed out that, although the axis of rotation of a rigid body passes through its center of gravity when acted upon by a couple, the axis of rotation and the axis of the couple are not necessarily parallel. For instance, a couple represented by A in Fig. 35 acting on the end of a free bar will cause it to rotate about the axis represented by B. Every rigid body has three principal axes of inertia to one of which the couple must be parallel in order that the axis of rotation and the axis of the couple shall be parallel. For the rod shown, one principal axis is parallel

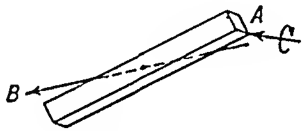


FIG. 35.—Torque Axis and Rotational Axis not Parallel.

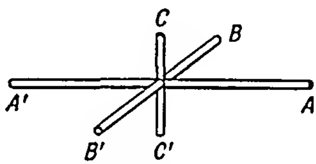


FIG. 36.—Equimomental Skeleton.

to the axis of the rod. The other two are perpendicular to it.

It can be shown that every rigid body can be represented by an equivalent skeleton of three rods, as shown in Fig. 36, so far as its inertia characteristics are concerned, called an equimomental skeleton. The three principal axes of inertia coincide with these three rods, the greatest inertia in this case being that about C, called I_C, and the least I_A, with I_B intermediate. If all three of these happen to be equal, as for a sphere or cube, the axis of rotation and the axis of an applied couple always are parallel.

For an excellent discussion of the principles presented in this chapter see reference 7.

43. Illustrations from Common Observation.—A complete discussion of these phenomena would take us far into rigid dynamics and can be found in works devoted to this

subject. For our purposes the outline in the preceding three sections is sufficient. The principles given are fundamental and not difficult to remember when once clearly seen. They are illustrated by many familiar phenomena. We know that when a body is dropped it falls in a straight line with no rotation. By definition, gravity acts through the center of gravity of a body. Thus the truth of the law given at the beginning of § 41 is at once seen.

A rotor in so-called dynamic unbalance will tend to wobble about its center of gravity owing to the centrifugal couple although its actual motion may be much modified by the elastic action of the bearing supports. This couple can be counteracted by a pair of balance weights producing an equal and opposite couple acting *anywhere* on the rotor, even out on the end of the shaft if there is room and if the shaft does not deflect appreciably because of the couple. This follows from the law given at the beginning of § 40. If a stick is thrown and rotated at the same time it will be seen to revolve about its center of gravity as nearly as can be observed, the center of gravity describing a smooth curve as though all the mass of the body were concentrated there.

If a button is spun on a string at high speed it will revolve about its center of gravity even if it is somewhat eccentric on the string.

If an unbalanced rotor on a flexible shaft is revolved at a speed far above its critical, its tendency to revolve about its center of gravity is so strong that the latter point remains practically stationary and the shaft is forced to revolve around it in spite of the elastic reaction of the shaft. The phase reversal which takes place in passing through critical speed is a result of the tendency for the center of gravity of the revolving system to remain stationary asserting itself against other forces. (See § 42.)

In vibration work, occasion sometimes arises to find the magnitude of the periodic force which may cause a rigid body to vibrate, such as the pump mechanism of an electric refrigerator. Suspension by long, flexible springs

will permit the body to vibrate as though practically free in space, when its vibratory motion can be accurately observed. If the body is not symmetrical it may be found to perform torsional vibrations about an axis not parallel to the axis of a periodic exciting couple. This, of course, is because the axis of the exciting couple is not parallel to a principal axis of the body.

Large errors have been made in estimating the magnitude of the stimulus causing a vibration, such as the amount of unbalance in a motor, by not suspending properly. If the motor is placed on a pad on an ordinary table, the latter may amplify the vibration amplitude several times, and vice versa. This subject is discussed further under the heading of elastic suspension. (See § 67.)

44. Application to Problems.—It follows from §§ 40–41 that the acceleration of the center of gravity of a rigid body is the same as if the translating forces acted at that point, and rotational acceleration takes place about the center of gravity as if the body were pivoted at that point, regardless of the actual translation which may be present. Thus, to find the motion of a rigid body acted upon by a given set of forces, translation and rotation may be treated separately.

First: set up force equations for acceleration of the center of gravity.

Second: set up equations for torque about that point.

In general, if all three dimensions are involved, there will be three equations of translation and three of torque, each of which may be independently set up.

Many practical problems can be set up so that there is one axis of translation and one of torque. If the axis of torque is parallel to a principal axis of inertia the rotation axis is parallel to the torque axis, and the problem is comparatively simple. Otherwise the torque must be

resolved into components parallel to the principal axes of inertia, so that there are two or three torque equations instead of one. Otherwise the procedure is the same. A torque can be represented by a vector parallel to its axis and resolved as though it were a single force.

Example 1.—Find the acceleration of a cylinder rolling down a plane inclined at an angle β (Fig. 37).

Let a = constant linear acceleration;
α = constant angular acceleration;
r = radius of cylinder;
W = weight of cylinder;
I_0 = moment of inertia of cylinder about its axis.

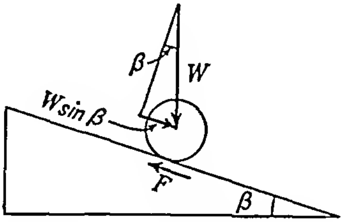


FIG. 37.—Cylinder Rolling Down Incline.

By Routh's rule (§ 26)

$$I = \frac{Wr^2}{2}$$

Let F = tangential force reaction on the cylinder from the plane. If a and α are positive for right-hand motion,

for translation

$$\frac{W}{g} a = W \sin \beta - F \quad . \quad . \quad . \quad . \quad . \quad (101)$$

and for rotation

$$\frac{Wr^2}{2g} \alpha = Fr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Since $\alpha = a/r$, from (102)

$$F = \frac{W}{2g} a$$

Substituting in (101),

$$\frac{3W}{2g} a = W \sin \beta. \quad \text{Therefore } a = \frac{2}{3} g \sin \beta \quad . \quad . \quad (103)$$

Example 2.—A cylinder of radius r_1 rolls back and forth on a concave surface of radius of curvature r_2. Find the frequency for an oscillation of small amplitude.

Using the symbols of previous problem with those of Fig. 38,
Translation

$$\frac{W}{g}\frac{d^2s}{dt^2} = F - W\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

since s is measured from AB, and θ is small, so $\sin \theta$ is set equal to θ.

Rotation

$$\frac{Wr_1^2}{2g}\frac{d^2\phi}{dt^2} = - Fr_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

since ϕ is measured from the middle position of the cylinder and the torque produced by F is thus negative.

Let us express ϕ and θ in terms of s. From the figure,

$$\theta = \frac{s}{r_2 - r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Also $(\phi + \theta)/r_2 = \theta/r_1$, from which

$$\phi = \frac{r_2 - r_1}{r_1}\theta = \frac{s}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Substituting (107) in (105) and solving for F,

$$F = -\frac{W}{2g}\frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

Substituting (108) and (106) in (104),

$$\frac{3}{2g}\frac{d^2s}{dt^2} + \frac{s}{r_2 - r_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

This represents a simple linear vibration as given by equation (3) whose frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{3(r_2 - r_1)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Example 3a.—Assume a cylinder to be removed from the earth so that it floats in space. How will it vibrate if a sinusoidal torque of maximum value L be applied to it parallel to its axis?

$$\text{Translation} = 0$$

$$\text{Rotation } \theta_0\left(\frac{-I\omega^2}{g}\right) = L$$

$$\theta_0 = -\frac{Lg}{I\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

where θ_0 = amplitude of the angular vibration about the axis of the cylinder.

The torque equation is written down directly, using the mechanical impedance $-(I/g)\omega^2$ for rotational oscillations in place of $-m\omega^2$ used for linear vibrations. (See § 31.)

The sign of θ_0 is negative with respect to the torque because the two are in phase opposition, as can be easily shown by experiment.

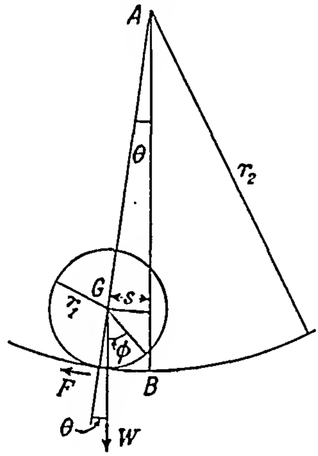


FIG. 38.—Oscillating Cylinder.

Example 3b.—Take the same problem as the previous except that the cylinder is restrained by a pivot at P with a tangential spring support as shown in Fig. 39, whose total elastic constant equals k. Otherwise the cylinder is free. Find the natural or resonant frequency. This problem differs from previous ones in that force of translation and also the torque each depend upon both the linear displacement x and the angular displacement θ, as shown in Fig. 40.

The point P is displaced by an amount $x + r\theta$ from its rest position P_0. The tangential force (taken horizontal since θ is small) is then equal to $k(x + r\theta)$.

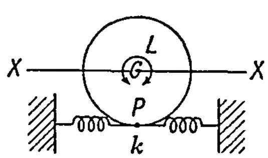


FIG. 39.—Oscillating Cylinder with Single Elastic Constraint.

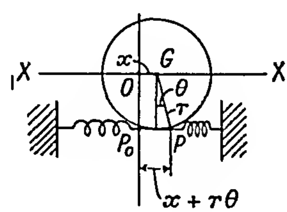


FIG. 40.—Same as Fig. 39 Showing Displacement.

Translation

$$(x + r\theta)k - x\left(\frac{W}{g}\omega^2\right) = 0 \quad . \quad . \quad . \quad . \quad (112)$$

Rotation

$$\left(\frac{x + r\theta}{r}\right)kr^2 - \theta\left(\frac{Wr^2}{2g}\omega^2\right) = L \quad . \quad . \quad . \quad . \quad (113)$$

Note that in rotation the elastic constant $L_1 = kr^2$, since it is defined as torque per radian. The angular elastic mechanical impedance is thus kr^2, and the linear as before is simply k. Also the angular mass impedance is $-(I/g)\omega^2$ and linear mass impedance is $(-W/g)\omega^2$.

In equation (113), since kr^2 is angular impedance, the displacement is expressed as an angle.

As before, the use of impedance requires that these be vector equations. Since no j terms are present the vectors are either in phase or differ in phase by 180°. They must therefore be along the same line.

Solving these equations by the usual method will give x and θ. Solving for θ,

$$\theta = \frac{L\left(1 - \dfrac{kg}{W\omega^2}\right)}{\dfrac{3}{2}kr^2 - \dfrac{Wr^2\omega^2}{2g}} \qquad (114)$$

The resonant frequency is that for which θ becomes infinite (and also x). This occurs when the denominator of (114) equals zero. Thus

$$\frac{Wr^2}{2g}\omega^2 = \frac{3}{2}kr^2$$

from which

$$f = \frac{1}{2\pi}\sqrt{\frac{3kg}{W}} \qquad (115)$$

This system has one natural frequency and thus one degree of freedom, which is evident from inspection.

CHAPTER VIII

CRITICAL SPEEDS OF ROTORS—RAYLEIGH'S AND MAXWELL'S THEOREMS

45. Critical Speeds of Rotors.—Consider a weightless shaft carrying a flywheel of weight W and supported in bearings at A and B (Fig. 41). The up-and-down vibration frequency is calculated as before for the case of a system of one degree of freedom (§ 10). Thus

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Furthermore, at the top and bottom of the swing the reaction due to acceleration just balances the elastic restoring force of the shaft—that is

$$\frac{W}{g}\omega^2 r = kr$$

In fact, this relation affords a method of directly calculating the frequency.

The equilibrium at this instant is just the same, however, as that which would exist all the time if the weight W were to whirl in a circle of radius r at the same frequency. In the first case, the motion is given by $y = r \sin \omega t$. In the second case, the circular motion is given by the simultaneous equations

$$\left.\begin{aligned} y &= r \sin \omega t \\ x &= r \cos \omega t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

When a rotor is revolving at such a speed, called a critical speed, a whirl is readily built up which may assume dangerous proportions if the centrifugal force is aided by a slight unbalance. If it is speeded up above the critical,

a change of phase takes place which is discussed in § 47, so that the rotor becomes stable again.

The frequency of whirl at which outward centrifugal force and restoring force exactly balance has been just shown to be identical to that for a linear vibration in a case like that in Fig. 41 for a perfectly balanced rotor.

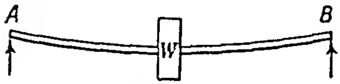


FIG. 41.—Shaft and Flywheel.

This same reasoning may be extended to the general case of a rotor of any mass distribution and stiffness. If its vibration frequency in a plane is found, its whirling speed or frequency is the same in all ordinary cases. It follows that the critical speeds of a uniform shaft can be found by the same analysis as that used for flexural vibrations of bars in § 38.

When once the transverse vibration frequencies of a rotor are found, however it be loaded, its critical speeds at which whirling may take place are approximately known.

46. Gyroscopic Action.—One effect, which enters for revolving rotors but is absent in a plane vibration, causes some difference between the linear and whirling frequencies in certain cases. This is a gyroscopic effect due to the angular momentum of the rotor.

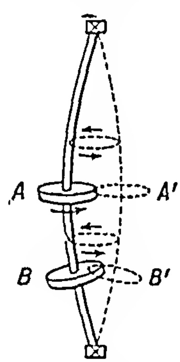


FIG. 42. — Gyroscopic Action of Flywheel.

This effect does not come in near the center of span of the rotor, where the shaft is always parallel to the bearing center line regardless of how much it deflects, as at *A*, Fig. 42. At *B*, things are different. The disk *B* has a so-called precessional motion forced upon it as the shaft whirls, which causes a restoring force additional to that due to the elastic stiffness, and thus produces a slightly higher frequency. Thus in a whirling at critical speed produced by unbalance the gyroscopic action tends to produce critical frequencies slightly higher

than frequencies due to linear vibrations. This effect is usually negligible and never larger, say, than 2 or 3 per cent.

A simple experiment shows very well how the gyroscopic force acts in this case. If a heavy top spinning at a high speed, at an angle of tilt such that it precesses slowly owing to gravity, is grasped by the upper end of its axis and an attempt made to hasten its precession, it will show a strong tendency to assume an upright position. This is just what happens with the disk B. When a precession is imposed upon it through the whirling motion of the shaft it tries to move inwards, thus producing a stiffening effect on the shaft.

It is to be noted here that the opposite action takes place if the precession of a top is retarded or if it is made to precess in the opposite direction. It tends to fall over.

On examining the direction of rotation of the top it will be found that forcing the precession in its direction of rotation causes the top to try to rise, and vice versa.

In our problem the direction of whirl and the direction of rotation of the shaft are always the same, so the first effect is present.

For a more complete mathematical treatment of this gyroscopic action see reference 8.

47. Phenomena at Critical Speed in Detail.—From the foregoing reasoning the critical speeds of shafts may be found, but in order to build up a whirl a stimulus of some sort must be present, as in the case of the building up of a linear vibration. The usual stimulus responsible for building up a whirl is caused by unbalance, that is, the center of gravity of the rotor is not exactly on the axis of rotation.

A detailed analysis will now be given of the simplest possible case where a flywheel of mass m is carried at the middle of the span of a weightless shaft. In order to eliminate the consideration of gravity, the shaft will be taken in the vertical position.

Gravitation has no effect on critical speeds, however, and in the case of a horizontal rotor it cancels out in the analysis.

Take a plane section through BC, Fig. 43, and take the XY plane in this plane. The origin O is taken on the center line between the bearings to which S, the geometrical center of the flywheel, will return if the shaft has no elastic deflection. The elastic deflection is given by r, and a is the distance between the geometrical center of the flywheel S and its center of gravity G. The distance a is actually very small, 2 or 3 mils, say, and equals zero

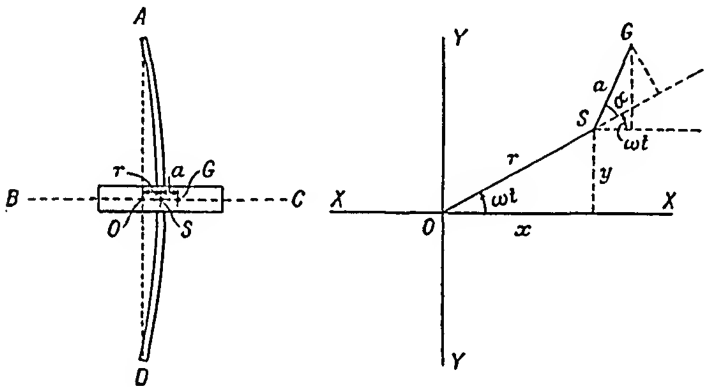


FIG. 43.—Unbalanced Flywheel on Revolving Shaft.

for perfect balance. It is a constant for a given amount of unbalance as is assumed in this analysis.

From Fig. 43 the acceleration of G is seen to be governed by the following two independent equations, one for the X component of motion and one for the Y component.

$$\left.\begin{aligned} m\frac{d^2}{dt^2}[x + a\cos(\omega t + \alpha)] &= -b\frac{dx}{dt} - kx \\ m\frac{d^2}{dt^2}[y + a\sin(\omega t + \alpha)] &= -b\frac{dy}{dt} - ky \end{aligned}\right\} . \quad (118)$$

where b is the friction or damping coefficient and k the elastic constant of the shaft.

These two equations are independent of each other, their solutions giving the X and the Y components of motion respectively. The first may be put in the form

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = ma\omega^2 \cos(\omega t + \alpha) \quad . \qquad (119)$$

Referring to Fig. 43 it is seen that the phase angle of r is taken as zero so that x has its maximum value when $\omega t = 0$.

The usual way of writing an equation of the type of (119) is to omit α, thus making the phase angle of the force $ma\omega^2$ zero instead of that of r. This causes r to lag behind $ma\omega^2$ by the angle α as appears from the solution of (120) given by (123). There is a transient vibration term and a steady forced vibration term, but since the transient soon vanishes it is only the forced vibration which is of interest. Writing the equation without α it has the form

$$m\frac{d^2y}{dt^2} + b\frac{dx}{dt} + kx = ma\omega^2 \cos \omega t \quad . \quad . \qquad (120)$$

Solving in terms of mechanical impedance, we have

$$x = \frac{ma\omega^2}{k - m\omega^2 + jb\omega} \quad . \quad . \quad . \quad . \quad . \qquad (121)$$

where x and $ma\omega^2$ are vectors. In like manner

$$y = \frac{jma\omega^2}{k - m\omega^2 + jb\omega} \quad . \quad . \quad . \quad . \quad . \qquad (122)$$

where the numerator contains a j factor because y must lead x by 90°, as can be seen from Fig. 43. Changing from the vector to the linear form of expression, as in § 28, these expressions become

$$\left.\begin{aligned} x &= \frac{ma\omega^2 \cos(\omega t - \alpha)}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \\ y &= \frac{ma\omega^2 \sin(\omega t - \alpha)}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (123)$$

where

$$\tan \alpha = \frac{b\omega}{k - m\omega^2}$$

These two expressions are 90° apart in both time phase and space phase, so that the motion is circular.

As previously pointed out, the resonant frequency of the whirl is the same as if it were a linear vibration, with the stimulating force acting in one plane.

48. Discussion and Interpretation of the Preceding Analysis.—Note that the preceding analysis assumes the frequency factor ω to be a constant. The whirling is considered as starting at a definite fixed ω. This solution does not cover the cases for speed rising up through the crittical or falling off through it, solutions of which would be very desirable for a more complete study. The differential equations for these cases have not been completely solved (reference 9), whereas for ω taken as a constant the solution is seen to be quite simple.

The physical interpretation of the whirling motion of such a shaft at various frequencies is very interesting. Note that for a given value of ω the angle α is fixed and OSG revolves about O as a unit. The term whirl as here used refers to a whirling motion of the shaft always coincident with the rotational speed ω, as indeed it must be, because the vibration is forced by the unbalance and therefore must be in period with it. There are other types of shaft whirling, where the whirling is always at resonant speed, regardless of the rotational speed of the shaft. These types of whirling are discussed in later sections (§§ 81 and 83). It is true, however, that whirling due to unbalance is severe only when the rotational frequency is in step with the resonant or natural shaft frequency, or very nearly so. This is well shown in the analysis just given. At low speeds, α is small, r and a being almost in line and r being comparatively small. As resonance is approached, r rapidly increases, as does α, until at the crit-

ical speed r is a maximum and α almost exactly 90°. As the speed rises still higher, r rapidly decreases and α approaches 180° so that G revolves inside instead of outside of S, and at very high speeds the system revolves almost exactly about G. At speeds high above the critical the tendency for the system to revolve about its center of gravity becomes so powerful that the stiffness of the shaft is almost completely overcome, so that S is forced about G as a center, just as a button spinning on a string will revolve about its center of gravity regardless of how the string passes through the button. When the shaft is running below its critical speed, centrifugal force which throws the center of mass outwards has the upper hand. When revolving above the critical speed the tendency for a revolving body to revolve about its center of gravity prevails. The critical speed is the transition point.

Interesting experimental studies of the vibration phenomena at this transition point have been made which verify the analysis of the preceding section (see reference 10).

49. Dunkerley's Formula for Critical Speed Frequency. —A very important early contribution to this subject of some forty years ago, which is a classic, was made by Dunkerley in England (reference 11). Experiments were made on model shafts carrying different numbers of flywheels and supported in different ways, which were checked against carefully worked-out analysis. The best-known result of this work is the following approximate formula for the critical speeds of shafts carrying a number of concentrated masses or flywheels.

$$\frac{1}{\omega^2} = \frac{1}{\omega_s^2} + \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} \text{ etc.} \quad . \quad (124)$$

where ω is the critical speed of the system; ω_s that of the shaft by itself; ω_1 that of flywheel 1 alone on the shaft, the latter considered as weightless; ω_2 that of flywheel 2 alone, etc. Although not exact, this formula is very useful in cases

where there are two bearings as shown in Fig. 44. For three-bearing sets the use of this formula is to be avoided, as it may involve a large error.

50. Energy Method—Rayleigh's Principle.—The usual way of calculating the frequency of whirl of a rotor is to employ the principle of conservation of energy. A plane vibration of some convenient amplitude is assumed, the maximum potential and kinetic energies estimated, and since these two must be equal to each other they are equated and the frequency constant ω is solved for.

To estimate these energies some form of deflection curve of the rotor must be assumed in which it vibrates, and the amount of the potential and kinetic energies will depend upon the deflection curve. It is at this point that an important general principle in vibration first proved by Lord Rayleigh will be employed. Rayleigh's principle may be stated as follows: *In any vibrating elastic system the displacement amplitudes are such that when the maximum potential and kinetic energies are equated to each other the lowest or gravest possible frequency is obtained.* If any other deflection curve than the correct one is taken, the frequency will come out a little too high. The system automatically deflects in such a way that its frequency is a minimum.

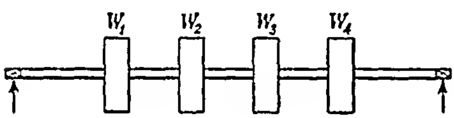


FIG. 44.—Shaft with Several Flywheels.

It can be seen that this principle must be true, since a deviation from the actual deflection curve of a vibrating system or shaft can be produced only by the presence of some additional elastic stiffness to maintain the assumed deviation. Removal of material to produce the deviation is not permissible because this implies removal of part of the original system assumed. Any additional stiffness thus produced, or restraint as it is called, increases the elastic potential energy and thus raises the vibration fre-

quency. For a rigorous analytical proof of this principle see Rayleigh's "Theory of Sound," Vol. 1 (reference 12).

Rayleigh also pointed out that if the static deflection curve of a loaded bar is used as an approximation instead of the actual vibration curve, in calculating the energies, the error in frequency will be small. This is true because considerable change in load distribution is necessary to produce appreciable change in the deflection curve, so that the curve for vibration load distribution in a bar or shaft is nearly the same as the curve for static load distribution. It will be recalled from § 38 that the loading of a uniform bar in producing a given deflection curve depends on the fourth derivative of that deflection curve, which explains the similarity of deflection curves for two quite different loadings of such a bar.

51. Practical Use of the Energy Method.—In nearly all practical work the static deflection curve of a rotor is used in calculating its critical speed. For instance, for the shaft shown in Fig. 44 carrying the four flywheels, if the weight of the shaft is neglected, the potential energy of bending due to gravitation equals

$$\tfrac{1}{2}W_1y_1 + \tfrac{1}{2}W_2y_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \qquad (125)$$

where y_1, y_2, etc., are the deflections due to gravity at W_1, W_2, etc. The maximum kinetic energy of a vibration of this same amplitude equals

$$\frac{1}{2}\frac{W_1}{g}\omega^2y_1^2 + \frac{1}{2}\frac{W_2}{g}\omega^2y_2^2 + \text{etc.} \quad . \quad . \quad . \qquad (126)$$

since

$$y = y_1 \sin \omega t; \quad \frac{dy}{dt} = \omega y_1 \cos \omega t, \text{ etc.}$$

Equating the potential to the kinetic energy and solving for $f = \omega/2\pi$

$$f = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{W_1y_1 + W_2y_2 + W_3y_3 + W_4y_4}{W_1y_1^2 + W_2y_2^2 + W_3y_3^2 + W_4y_4^2}} \qquad (127)$$

In the case of rotors with distributed loads the load deflection curve is found by graphical statics, the shaft being arbitrarily divided into a number of sections for this purpose. The weights of these sections and their corresponding deflections are used as just shown, giving the general formula

$$f = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{\Sigma Wy}{\Sigma Wy^2}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (128)$$

where the summation sign Σ is used for brevity.

When required, a more accurate deflection curve can be found by using the static load-deflection curve to calculate a second one where the acceleration reaction of each section ($W\omega^2y/g$) is used in place of its weight. Any convenient value of ω is assumed, and the values of y are taken from the first curve. By Rayleigh's principle the frequency thus found will be slightly lower than that which would be obtained by using the original static deflection curve to estimate the energy.

In three bearing sets this process may fail in that the successive approximations give new frequencies which diverge from the true or gravest frequency rather than approach it (see § 55).

52. Elasticity of Bearings and Critical Speeds.—Thus far, perfectly rigid bearing supports have been assumed in critical speed calculations. Actually, this may be far from the truth, especially for large, heavy rotors. For every rotor there is a certain degree of elasticity in the bearings themselves, which lowers the critical speed below that for rigid bearings. This lowering of critical speeds is frequently as much as 25 per cent and may in extreme cases be over 50 per cent; that is, the critical speed may be reduced to less than one half that for perfectly rigid bearings. Furthermore, bearing pedestals will normally deflect sideways more easily than up and down, with the result that a rotor may have two distinct critical speeds, one of them a sideways vibration and the other up and down, the

sideways vibration being the lower of the two, perhaps 10 or 15 per cent lower than the other. The actual motion in such cases is not linear but elliptical in character, with the major axis several times the minor.

In critical speed calculations the effect of the yielding of the bearings may be taken account of by estimating their elastic deflections a and b as produced by the bearing reactions during vibration, and measuring the displace-

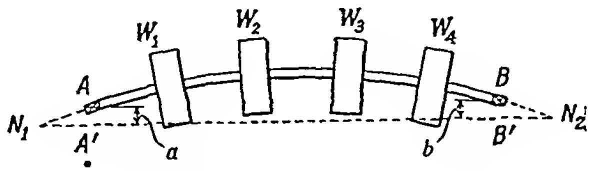


FIG. 45.—Effect of Elasticity in Bearings.

ments y_1, y_2, etc., from the line $A'B'$ instead of AB as shown in Fig. 45. If the line of the shaft axis is extended from the bearings A and B, it will intersect $A'B'$ extended at N_1 and N_2, which points may be considered the nodes of this type of vibration. The more the bearings yield, the farther apart are N_1 and N_2, and thus the lower the frequency.

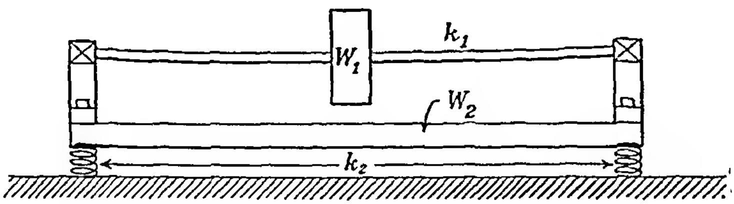


FIG. 46.—Rotor with Elastic Base Supports.

53. Effect of Elasticity and Mass in the Base.—If the rotor is mounted on a platform or structure which is itself not sufficiently rigid, critical speeds will enter which are characteristic of a compound vibrating system. This condition is shown in idealized form in Fig. 46. The weights and elasticities of the system are marked to correspond with that analyzed in § 33, the stimulus, in this case, coming from unbalance. As in the former instance,

the system has two natural frequencies: the lower where W_1 and W_2 vibrate in phase; and the higher where they vibrate against each other, 180° out of phase. The latter is likely to produce the sharpest shaft deflection, although the possibility of its presence is often overlooked in design as is also the compound character of the vibrating system. In studying the effect of mass of and elasticity in the base on these two critical speeds it is instructive to plot curves showing the variation of the critical speeds with variation of the ratio W_2/W_1 and k_2/k_1.

Thus far, only the up-and-down criticals have been considered. If the base is practically rigid sideways, only the regular sideways shaft critical speed is present in addition, making three critical speeds in all.

54. Upper Critical Speeds.—Every shaft has a series of critical speeds, of which the first or fundamental is much the most important. A continuous elastic rotor like a shaft has an indefinitely large number of higher critical speeds according to ideal theory, but in practice only the first two or three are of importance. Those higher than this are relatively difficult to excite, and are almost always well above operating speed. Upper critical speeds have not been mentioned thus far except for the compound system of the previous section.

The characteristic deflections and frequencies for the more important critical speeds of a uniform shaft supported in two and in three equally spaced bearings will now be compared.

Figure 47 shows the deflection curves of a uniform shaft supported between two bearings for the first and second critical speeds. These deflection curves and their frequencies are found by the same analysis as that given for flexural vibrations of bars presented in § 39. The curves in this case are of sinusoidal form, the first critical having one loop with nodes at the bearings, the second two loops with a node half way between the bearing nodes, the third with two nodes equally spaced between the bearings,

etc. The frequency is in each case proportional to the square of the number of loops, that is, the second critical is 4 times the first, the third 9 times the first, etc.

For a uniform shaft supported in three equally spaced bearings the first three critical speed deflection curves are as shown in Fig. 48.

In this case, the first critical speed is the same as for

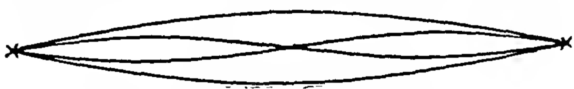


FIG. 47.—Critical Speed Vibrations of Shaft.

a two-bearing set of half the length of the three-bearing set, as though the shaft were cut at the middle bearing. This must be true because the middle bearing is at a node where both the flexure and the reaction due to vibration are zero.

The second critical is a direct result of the presence of the middle bearing which supports a vibration reaction and at which there is a point of maximum flexure.

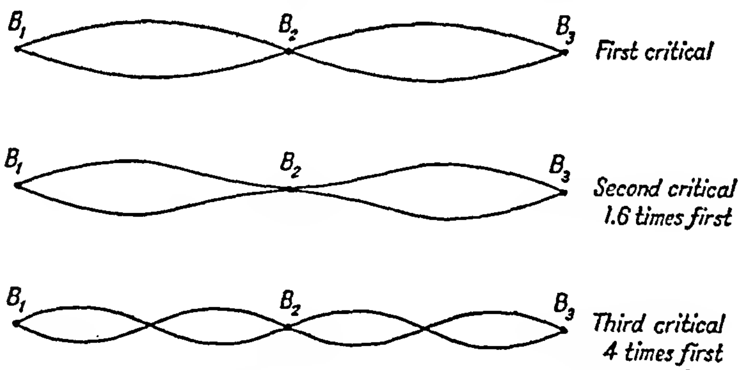


FIG. 48.—Critical Speed Vibrations with Three Bearings.

The third critical corresponds exactly to the second critical for a single span just as the first corresponded to the first for a single span, as previously noted. Like the first critical speed, the frequency of the third would not be altered if the shaft were cut at the middle bearing. Thus the frequency of the third critical is 4 times that of the first, but that of the second about 1.6 times the first.

The frequency of the second critical for such a shaft in three bearings can also be calculated from the analysis of § 39, using the conditions at the middle bearing in addition to the ends in eliminating the constants in equation (91). For instance, referring to Fig. 48, at the middle bearing $u = 0$ and the slope $dy/dx = 0$. The slope, however, would not be zero at the middle bearing if the spans were not of equal length, so the solution then would not be so simple.

55. Practical Application.—The ideal cases presented in the previous section are useful for illustration but are seldom realized in practice for several reasons. The mass distribution of a rotor is not uniform, but is concentrated along the middle part of the span. For two-bearing sets and also for three-bearing sets this has the effect of bringing the criticals previously discussed closer together. For relatively heavy flywheels and light shafts they may be quite close together. Three-bearing sets have the complication that unbalance causing vibration may be present in both spans and not in the same plane. For this reason three-bearing sets are often troublesome to balance. With four bearings it is sometimes possible to shift the rotor of one span with respect to the one in the other by uncoupling and coupling again at a different angle so as to bring the unbalance into the same plane, and thereby facilitate balance. The two middle bearings of a four-bearing set are usually so close together that such a set acts like a three-bearing set as regards critical speeds.

In such cases, the energy method is used to find the critical speeds as presented in § 51, but for three-bearing sets the true deflection curve is often difficult to establish even approximately, and errors must be looked out for. Nearly all the vibration amplitude may be present in one span if the span is long or carries a particularly heavy rotor. Figure 49 shows a typical case.

56. Reciprocal Theorem.—A very interesting general theorem in elasticity first published in simple form by

Maxwell in the *Philosophical Magazine* about seventy years ago (reference 13) will now be introduced because, in addition to its general interest, it is sometimes used in calculating the critical speeds of three-bearing sets to find the reaction on the middle bearing.

If two separate points on any elastic body be selected and a given weight placed at the first point, the deflection of the second point thus produced is the same as the deflec-

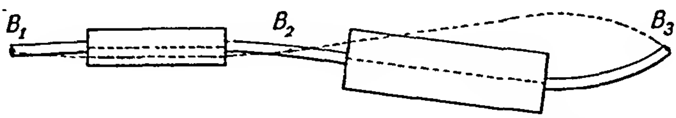


FIG. 49.—Unsymmetrical Vibration at Critical Speed.

tion produced at the first point on placing the weight on the second point. For instance, a force F acting at P_1 on the cantilever shown in Fig. 50 produces the same downward deflection at P_2 as that produced at P_1 when the force is removed from P_1 and acts at P_2.

The proof of the reciprocal theorem for the particular case of this cantilever of Fig. 50 will now be given.

Let y_1 and y_2 be the deflections produced at P_1 and P_2 when F is applied alone.

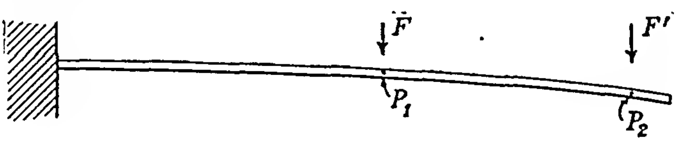


FIG. 50.—Reciprocal Theorem.

Let y_1' and y_2' be the deflections produced at these points when F' is acting alone.

The potential energy of bending produced by first applying F by itself and then F' in addition evidently equals that produced by applying the weights in the reverse order, but at the same points as before, because the final deflection in each case is produced by the same two forces acting in the same positions.

Expressing the energy mathematically

$$\overbrace{\tfrac{1}{2}Fy_1}^{\substack{F \text{ alone} \\ \text{at } P_1}} + \overbrace{\tfrac{1}{2}F'y'_2 + Fy'_1}^{\substack{F' \text{ in addition} \\ \text{at } P_2}} = \overbrace{\tfrac{1}{2}F'y'_2}^{\substack{F' \text{ alone} \\ \text{at } P_2}} + \overbrace{\tfrac{1}{2}Fy_1 + F'y_2}^{\substack{F \text{ in addition} \\ \text{at } P_1}} \quad (129)$$

Note that in applying the second force, in each case the first force already acting in full does additional work. Cancelling off identical terms from each side of the equation gives

$$Fy'_1 = F'y_2 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

This is a mathematical statement of the reciprocal theorem for the case of the cantilever of Fig. 50.

If F and F' are equal, as is the case when the same weight is used in each position as assumed in the first illustration,

$$y'_1 = y_2$$

which shows that if the same weight is used in each position the two alternate deflections are exactly equal.

The proof given is seen to assume that the deflections produced on adding a force are the same, whether or not initial deflections, due to some other force, already exist. This illustrates another fundamental law of elasticity which follows from Hooke's Law. It is called the principle of superposition of strains and holds to a close approximation provided the strains are not so great as to materially change the shape of the body, such as an excessive downward bending of the cantilever or the large deformations of a rubber body. The whole mathematical theory of elasticity assumes the truth of the principle of superposition of elastic strains, whereby a given loading produces its own particular strains on top of strains already present from previous loadings as though these initial strains did not exist.

57. Application to Three-bearing Sets.—As an example, assume a three-bearing set loaded as shown in Fig. 51. The problem is to find the middle bearing reaction R_B.

First, calculate the deflection y'_B for a load L at B with the bearing removed. Then, from Hooke's Law that strain is proportional to load, if R_B produces the deflection y_B (with the bearing B removed), it follows that

$$\frac{R_B}{L} = \frac{y_B}{y'_B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

In this relation, if y_B is known, R_B can be calculated, since L and y'_B are known. Now y_B can be found from Maxwell's theorem as follows. Again, with bearing B removed, if L produces deflection y_1 at 1, y_2 at 2, etc., unit load at B produces deflections y_1/L, y_2/L, etc., respectively.

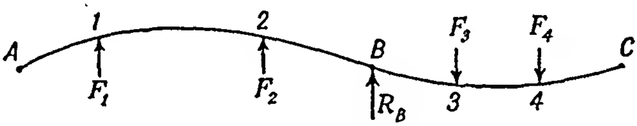


FIG. 51.—Forces on Shaft of Three-bearing Rotor.

By Maxwell's theorem, unit load at 1 produces deflection y_1/L at B; at 2 it produces y_2/L at B, etc. It follows that load F_1 at 1 produces F_1y_1/L at B, and load F_2 at 2 produces deflection F_2y_2/L at B, etc. Thus the total deflection at $B = (\Sigma Fy)/L$. But this equals y_B, since the bearing load just compensates this deflection, that is

$$y_B = \frac{\Sigma Fy}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Substituting (132) in (131),

$$R_B = \frac{\Sigma Fy}{y'_B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

where F has the values F_1, F_2, etc., and y the values y_1, y_2, etc., produced by L at these points, and y'_B is the deflection L produces at B. The problem is thus solved by calculating the deflection curve between A and B for a single arbitrary load at B and by applying Maxwell's theorem.

CHAPTER IX

TURBINE WHEEL VIBRATIONS

58. Flexural Waves in Turbine Wheels.—During the few years before 1920, enormous development took place in steam turbines, which has indeed continued since then, but about that time a number of disk wheel failures took place which were not understood. An investigation covering a period of two years proved beyond doubt that these failures were due to fatigue cracks produced by a particular type of axial resonant vibration. When the presence of this resonance was once demonstrated by tests, its cause was easy to picture, as is often true in such cases.

It is well known that sound waves travel with a definite velocity characteristic of the density and elasticity of the medium which transmits them. In like manner, the edge or periphery of an elastic disk will transmit a train of flexural waves around it with a characteristic speed determined by the mass per unit area, and the stiffness of the disk. They differ from sound waves in that they involve a bending or flexure of the elastic material instead of direct compression and rarefaction. Since the boundary of the disk has a finite circumferential length the wave lengths must have certain fixed values. For instance, the disk edge may carry a continuous train of 2, 4, 6, or 8, etc., waves whose wave length is $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, etc., of the circumferential length of the disk. Furthermore, instead of all wave trains moving with the same characteristic velocity, as in sound, each of these wave trains has a different velocity, the longer wave lengths being transmitted with considerably higher speeds than the shorter.

59. Critical Speeds of Turbine Wheels.—It was found that when the forward rotational speed of a turbine wheel

was exactly the same as the characteristic speed of one of the wave trains a dangerous condition existed because the wave train was easily built up by a comparatively small excess of axial pressure on some part of the wheel. The wave train always moved along the disk edge against the direction of rotation of the wheel in such a way that it remained stationary in space owing to the fact that its backward speed was the same as the forward rotational speed of the wheel. A little consideration will show that a single pressure spot on the wheel will build up such a wave train if it acts in the right phase relation with the wave. If in Fig. 52 the bottom arrow represents the wheel motion and the top the wave motion of equal velocity, the wave shown will remain stationary on the diagram as the wheel edge moves to the right. A fixed force at P acting

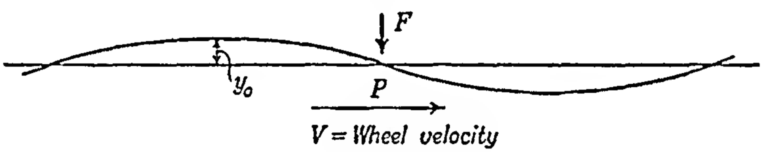


FIG. 52.—Wave in Turbine Wheel.

between the crest and trough of the wave will continually supply energy to the wave train. The rate of energy supply equals the work per second which equals F times the transverse velocity component in the direction of F. Let the wave equation be

$$y = y_0 \sin \omega t = y_0 \sin 2\pi f t$$

where f = frequency.

If λ = the wave length, $f = V/\lambda$, so that

$$y = y_0 \sin \frac{2\pi V t}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

This equation gives the transverse vibratory motion of a particle of the wheel as the particle moves to the right against the standing wave. The transverse velocity of the particle $= dy/dt = 2\pi y_0 V/\lambda \cos 2\pi V t/\lambda$. As the particle

passes through the fixed point P it has its maximum transverse velocity $= 2\pi y_0 V/\lambda$.

The force F at P acts continually on a succession of such particles, all of which have the transverse velocity $2\pi y_0 V/\lambda$. Thus

$$\text{Work per second} = \frac{2\pi F y_0 V}{\lambda} \quad . \quad . \quad . \quad (135)$$

This rate of energy supply is easily found for a given velocity of motion V, wave length λ, and force F, for any amplitude y_0.

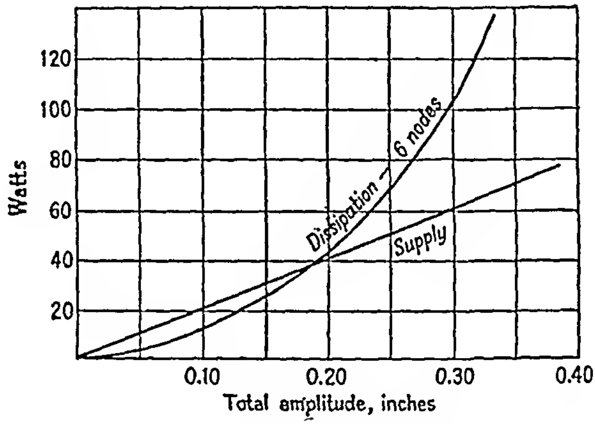


FIG. 53.—Energy Supply and Dissipation in Vibrating Turbine Wheel.

Calculation shows that, if F is even a few pounds in magnitude, long waves on a large, high-speed wheel are easily built up.

The curves of Fig. 53 show the energy supply curve and the energy dissipation curve as dependent on amplitude for a typical case, in watts. The wave amplitude builds up until the vibration dissipating power of the disk equals the energy input, where the curves intersect.

Note that the ability of such a large steel wheel to dissipate energy is comparatively small, and that only a few watts of energy input may produce a dangerous vibration. The dissipation curve shown was obtained from a direct

measurement. The input curve is based upon formula (135), assuming reasonable values of F and V. Note also that the input curve is linear, but the dissipation curve shows an increasing rate of dissipation with amplitude. Thus at very small amplitudes the probability of a wave building up is large, since the input curve is well above the dissipation curve in this region.

60. Prevention of Turbine Wheel Vibration.—Such a critical speed vibration of a turbine wheel is a serious danger even if the wave amplitude be comparatively small, because of the possibility of a fatigue crack developing in the wheel from the periodic stresses present, often resulting in failure if not discovered in time. It is now known that many steels are much more susceptible to fatigue failures in the presence of steam at high temperatures, which must be taken account of in turbine wheel design. The allowable stress under these conditions is many times less than that permissible in a structure under static load.

Just as in shaft and rotor design, the means employed to prevent turbine wheel vibration is to make sure that the critical speeds are different from the running speed. The problem is much more difficult in the case of turbine wheels, however, because the dangerous critical speeds are numerous and hard to avoid. In very large turbine wheels it is impracticable to design them with all critical speeds above the running speed because of their large size and mass, so the critical speeds are sometimes spaced on both sides of the running speed.

The flexural wave trains responsible for the critical speeds of course traverse the wheel edge in its completely bucketed condition, the wheel with its buckets vibrating as a unit. The most dangerous critical speeds are those involving two and three waves or the so-called four- and six-node criticals, because the waves are long and extend radially inwards below the bucket zone, so that the maximum flexures are in the wheel itself, involving the danger of wheel failures. Where the number of waves in the train

is large, only the buckets are apt to be involved, the greatest flexures occurring in the bucket zone, with danger of bucket failures. Many obscure cases of bucket failures have been traced to a train of many short waves carried along in the bucket zone well outside of the wheel proper. In such cases there may be as many as eight or ten waves in the entire circumferential train.

Although the wave velocities are susceptible to accurate calculation from theory, it has been found that, for safety, tests and frequency measurements must be made on revolving wheels on a test shaft in a steam box. Theory does not take account of conditions of initial stresses, which are present to a greater or less extent in the best turbine wheels. It is easily demonstrated by test that a thin disk with initial hoop tension or compression along its edge will vibrate at radically different frequencies from those of an ideal stress-free disk.

The effect of centrifugal force is to raise the flexural stiffness of the disk and increase the wave speed, but this effect is appreciable only for the longer waves, and furthermore it is easily calculated. See reference 14.

CHAPTER X

BALANCE OF RIGID ROTORS

61. Unbalance in a Rigid Rotor.—If the axis of rotation of a rigid rotor passes exactly through a principal axis of inertia (§ 42), its rotation will be perfectly smooth; but if the axis is slightly displaced from this ideal line, periodic bearing reactions in step with its rotational frequency will result. Figure 54 shows a cylindrical rotor whose principal axis about which rotation should take place for perfect *balance*, as it is called, is supposed to exactly coincide with the axis of the cylinder. Unbalance is assumed, however, in that the axis of rotation is displaced from this ideal line

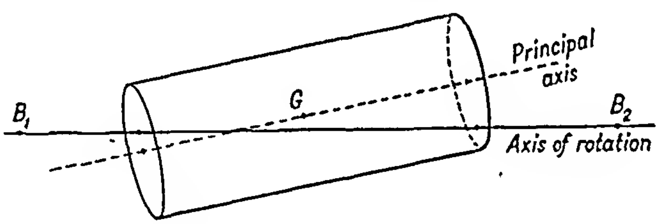


FIG. 54.—Example of Unbalanced Rotor (Exaggerated).

as shown exaggerated in the figure. This illustrates the most general case of unbalance, where the rotational axis in its failure to coincide with the principal axis of the rotating mass does not even pass through its center of gravity. Practically all manufactured rotors, large and small, show more or less of this general type of unbalance, a very small amount of which causes vibration, especially in high-speed rotors. Regardless of precision in manufacture, slight irregularities in mass distribution are found to be present, sufficient to cause perceptible vibration, even though the shaft passes exactly through the geometrical axis of the rotor.

That part of the unbalance due to the axis of rotation not passing through the center of gravity of the rotor is termed *static* unbalance, because it can be detected by a delicate static test. If the rotor is placed with its bearings on smooth, level ways it will roll under the influence of gravity until its center of gravity is in the downward position. That part of the unbalance which remains due to a slight lack of alignment of the principal axis and the rotational axis is termed *dynamic* unbalance, because it can be detected only by a rotational test bringing into play the dynamic forces of rotation, or the centrifugal forces.

62. Analysis of Unbalance of a Rigid Rotor.—It might be gathered from the preceding section that two steps are

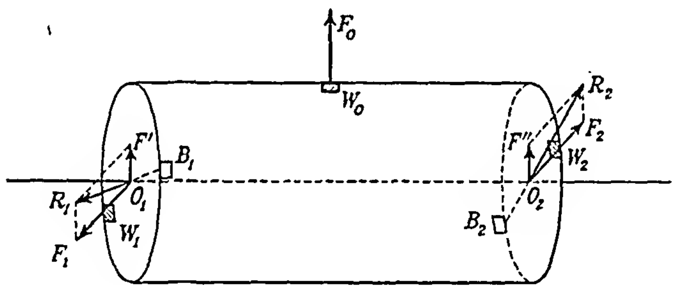


FIG. 55.—Complete Balance Established by Two Weights.

necessary in the balance of a rotor, one for the static and one for the dynamic unbalance. This is not so, however, as all unbalance present must show up in a rotational test. It will now be shown that only two balance weights are necessary to correct the unbalance of a rotor in the most general case.

The rigid rotor in Fig. 55 illustrates such a general case of unbalance. The center of gravity is displaced from the axis of rotation by the central weight W_0 opposite the center of gravity while at the same time a couple is produced by the pair of weights W_1 and W_2 in a different longitudinal plane from W_0. It will be recalled that a couple

tends to make a free body turn about its center of gravity regardless of where the couple is applied (§ 40). Thus an unbalanced couple in the axial plane of a rotating body, regardless of where the couple acts, may be represented by W_1 and W_2 as shown in the sketch. The force vectors represent the reactions produced by the weights during rotation. The force vector F_0 is replaced by its equivalent of two component vectors F' and F'' in the transverse planes of F_1 and F_2 respectively. The vectors F' and F'' are the equivalent of F_0 if they have equal and opposite moments about the center of gravity of the rotor and at the same time the sum of their magnitudes equals that of F_0. If the resultant vectors R_1 and R_2 are each exactly counteracted by the balance weights B_1 and B_2, complete balance will be established. Any case of an unbalanced rigid rotor may be represented in this way.

Note that W_0 produces the static unbalance, and W_1 and W_2 produce the centrifugal couple which is the cause of dynamic unbalance and can be detected only by a rotational test. Note also that a rotational test is sufficient to detect all unbalance in the rotor.

63. Methods of Balance.—Modern types of balancing machines employ what may be called the double fulcrum method. The rotor is elastically supported in a light, stiff frame which is first pivoted in the plane of B_1 while weight B_2 is applied, and then pivoted in any other plane while weight B_1 is applied (§ 62). By this method the addition of B_1 has no influence on the compensating action of B_2 because when the frame is pivoted in the plane of B_1 the moment of that weight about the pivot is zero. When the moments about two separate planes are at the same time zero, no moments tending to vibrate the rotor can be present, and complete balance is thus established. Figure 56 shows the plan of a simple outfit for balancing small rotors, using the traverse method. The rotor is placed in the frame F, which for small rotors may be of wood, as such a frame can be made light and stiff. The

rotor R rests in half bearings in the frame and is driven by the motor M through the belt L whose tension is adjusted by the screw S by which the motor on its platform P can be slid to and from the rotor. The rotor is first pivoted in the plane C, say, by the adjustable pivots, those in the plane D being withdrawn. A spring, not shown, is placed under the frame at the bearing B, of sufficient stiffness to cause resonance at about 700 or 800 r.p.m. The rotor is driven up through this resonant frequency, and the deflection indicator E is adjusted with its button resting on the top of the frame so that it registers the vibration present. The arc through which the needle oscillates is an approximate

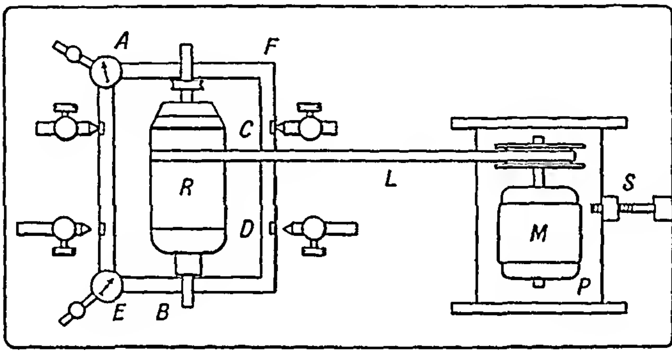


FIG. 56.—Balancing Set.

measure of the unbalanced moment about the fulcrum in the plane C. The indicator should read down to mils (thousandths of an inch) at least. A standard Starrett indicator serves well.

To find the amount and position of the weight the vibration amplitude is first read and then a trial weight of small size is successively placed in four positions 90° apart in the plane D, the amplitude of vibration shown by the indicator being read in each case. A curve of rise and fall of amplitude as dependent on the angular position of the trial weight is then plotted, from which the amount and position of the weight can be estimated. An inspec-

tion of the curve shows in what position the weight subtracts most from the unbalance and where it adds most to it, the former position being where the balance weight belongs. Its amount can be estimated by noting the change of vibration amplitude produced by the weight in different positions and comparing with the original vibration amplitude present, assuming that the vibration amplitude is a direct measure of the weight required.

A piece of wax may be used as the trial weight for small rotors. Besides having a light frame to insure large amplitudes, the outfit must not vibrate as a whole because of insecure foundation, nor must it be placed where floor vibrations are transmitted to it, otherwise inconsistent readings will be obtained on the indicators. A good way to isolate the apparatus from such vibrations and prevent detrimental shaking is to place the outfit on a massive iron plate which rests on toy rubber balls or on an inflated inner tube of an automobile tire. Also, the driving motor must be well balanced, and a light belt used to give a constant tension, which must be sideways, that is, its pull should be perpendicular to the oscillatory motion of the frame, nor should the pull be too great.

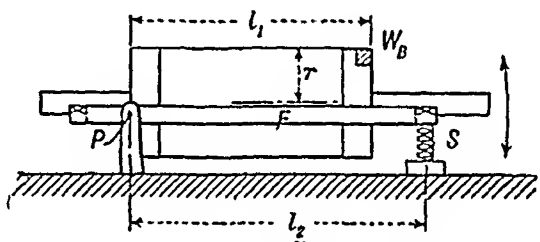


FIG. 57.—Rotor in Balancing Frame.

To balance the other end of the rotor, the process is repeated, using the other fulcrum. The weights thus determined should produce complete balance so that no oscillatory motion is shown on the indicators. When the process is thoroughly understood a little practice will insure great proficiency on the part of the operator, both as to speed and as to accuracy.

64. Analysis of Frame Vibration.—In Fig. 57 the frame F is shown pivoted at P and carrying a rotor. Let us find

the vibration produced by an unbalance represented by the weight W_B. Let the moment of inertia of the frame and rotor about $P = I$. Torque amplitude about P due to W_B is equal to

$$L = \frac{W_B \omega^2 r l_1}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

If k = elastic constant of the spring S, the torque per radian of arc about $P = L_1 = kl_2^2$, since l_2 is the lever arm, and for a radian of motion the force on the spring would be kl_2; and L_1 = force times lever arm. Thus

$$L_1 = kl_2 \times l_2 = kl_2^2 \quad . \quad . \quad . \quad . \quad . \quad (137)$$

The amplitude of the angular oscillations about P is found by the mechanical impedance method in a way perfectly analogous to that used in the case of a linear vibration where F is replaced by L, m by I, k by L_1, and the displacement y by the angle θ. Thus

$$\theta \times \text{impedance} = \theta\left(L_1 - \frac{I\omega^2}{g}\right) = L$$

$$\theta = \frac{L}{L_1 - \dfrac{I\omega^2}{g}} = \frac{\dfrac{W_B}{g}\omega^2 r l_1}{kl_2^2 - \dfrac{I\omega^2}{g}} \quad . \quad . \quad . \quad (138)$$

If ω is large so that kl_2^2 is relatively a small term, it may be dropped so that

$$\theta = \frac{-W_B r l_1}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

where the minus sign shows θ and the exciting forces are in phase opposition, and may be dropped in considering magnitudes only.

It is seen from this expression that the angle θ through which the frame oscillates is proportional to the amount of unbalance present, as assumed in § 63. Furthermore,

the amplitude angle is inversely proportional to the moment of inertia of the system about P so that to make the indicator readings as large as possible in order to detect small amounts of unbalance the frame must have the smallest possible inertia consistent with stiffness. Reasonable stiffness of frame is essential, since deflections in the frame disturb the indicator readings.

Many of the older commercial balancing machines have

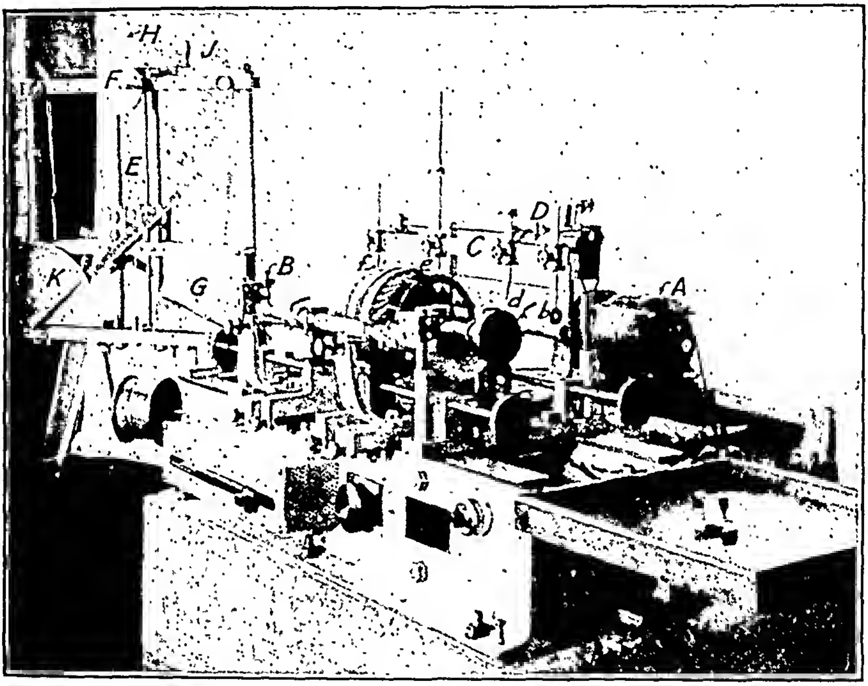


FIG. 58.—Thearle Balancing Machine.

frames far too heavy for accurate work, because of their large inertia compared with that of the rotor.

65. Thearle's Automatic Balancing Machine.—A very interesting and efficient type of balancing machine has been worked out by E. L. Thearle, of the General Electric Company, the theory and operation of which are described in a paper presented before the Applied Mechanics Division of the American Society of Mechanical Engineers in June, 1931 (reference 15). A brief description of it will be pre-

sented in this section. The double fulcrum method is used as described in § 63. Figure 58 shows a photograph of the machine. Balance is accomplished by two weights just as in the apparatus shown in Fig. 56, but the amount and position of each balance weight is found from a single observation of a special automatic balancing head (*d*, Fig. 58, and Fig. 59).

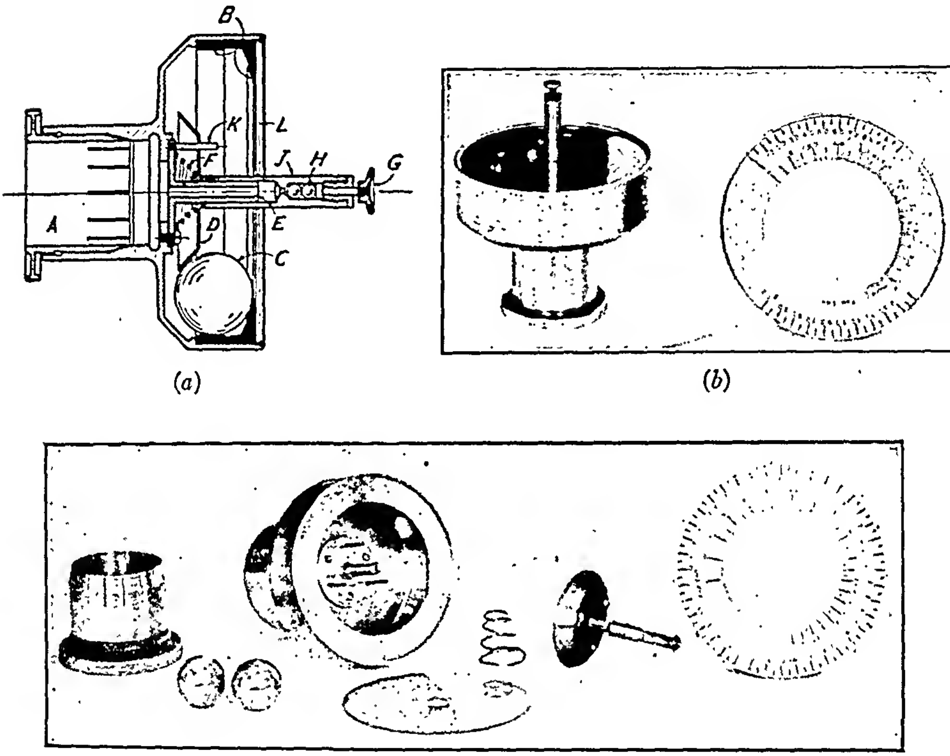
Referring to Fig. 58, the rotor is seen to be mounted in a frame which is made up of a pair of light steel tubes with adjustable aluminum cross-members in which half bearings are mounted. The whole frame is carried on a four-point spring suspension supported by the curved uprights shown. A pair of adjustable fulcrums are provided, consisting each of a pair of phosphor bronze springs held at their lower ends in a pair of vises, one pair for each supporting plane, which can be clamped or released at will. Three of these vises, and also their clamping rods with handles at the ends, appear in the figure. Either pair of fulcrums is thus readily put into action or released by a twist of the proper pair of handles. The position of these fulcrums on the frame is also adjustable, depending on the size of the rotor and the required balance weight planes.

The automatic balancing head is shown in Fig. 59. The collar *A* is clamped over the end of the motor shaft by giving it a half turn, this fixing the head in position. The rotor is then driven up through the critical speed of the balancing frame so that it seeks its own axis of revolution (§ 43).

The balancing head is then whirled bodily by the shaft in such a way that, when the steel balls *C* are set free in their race by a pressure on *G*, releasing the clamp *D*, they immediately roll to a position in the race opposite to the heavy spot on the rotor, thus cancelling the effect of unbalance, and reducing the vibration to zero. If they roll too far the rotor becomes unbalanced the other way, inducing, in turn, a whirling vibration of the head such that the steel balls will tend to move back part way, etc., until they

reach a position of stable equilibrium when perfect balance is attained such that the head spins exactly about its axis.

Note that operation above the critical speed of the frame is necessary; otherwise the rotor vibrates with its heavy side out, so that when the balls are released they



(a) (b)

(c)

FIG. 59.—Automatic Balancing Head.

assume a position which makes the unbalance worse instead of improving it.

In counteracting the unbalance, the steel balls assume positions such that the vector sum of the centrifugal forces produced by them just counteracts the unbalance, being nearer together or farther apart according as the unbalance is large or small. With no unbalance they take positions opposite each other. If the unbalance is excessive they fail to eliminate unbalance even if they come together.

Nevertheless, the plane of unbalance is indicated, so that a trial balance weight can be attached which so reduces the unbalance that a second trial will give complete balance. From the positions of the steel balls the amount of unbalance is determined in both phase and magnitude, and a correction weight can be easily calculated and placed on the rotor.

Balance accomplished by machines of this type is rapid and accurate, as has been proved by their use in production.

For very small or very large rotors, other methods of balance are desirable. For small rotors the manufacture of the head becomes a watchmaker's job; furthermore, its mass becomes so great compared with that of the rotor that sensitivity is reduced.

For large rotors of over a ton in weight other methods such as that of the Akimoff type of machine require a less expensive outfit and one which is sufficiently rapid for massive rotors which are not balanced in quantity.

CHAPTER XI

PREVENTION OF NOISE AND VIBRATION THROUGH ELASTIC SUSPENSION; THEORY OF VEHICLE SUSPENSION

66. Noise Elimination Classification.—An important field under the subject of vibration prevention is the elimination of objectionable noises and hums which have their origin in vibrating bodies. Sound itself is a type of vibration, and we might go far into the field of acoustics in noise elimination work. In this discussion it will be touched upon only as far as it is necessary to understand the principles at the basis of noise and vibration prevention through elastic suspension.

Noises from machinery, from the standpoint of elimination, fall into two classes:

(1) Those which are radiated directly into the air from parts of the apparatus itself.

(2) Those which do not radiate in this way, but are transmitted through the foundations to the building structure and appear with objectionable intensity often at remote parts of the building such as walls and panels which may resonate to the exciting frequency.

Preventive methods may also be classified as follows:

(1) Prevention at the source by study of the design.

(2) Elimination through screening.

(3) Elimination through elastic suspension.

The first of these measures is always aimed at but, even if at all possible, may require expensive investiga-

tions and more knowledge of design than is at present possessed.

Of the two remaining preventive measures it is evident that air-borne noises are not to be eliminated by elastic suspension, but that sound-absorption screens and filters must be resorted to.

Elastic suspension is a specific for noises transmitted through foundations, and this is the phase of noise elimination which will be treated in the few sections which follow.

In general, air-borne noises are of comparatively high frequency, say 500 cycles per second and above, whereas foundation hums are seldom of frequency above 500 cycles, and are usually considerably below this figure, and may consist of very low frequencies indeed such as those arising from unbalance of rotating bodies.

67. Theory of Elastic Suspension.—The purpose of elastic suspension is to prevent the transmission of vibrations to the floor or foundation on which the vibrating body is placed. In order to specify the effectiveness of an elastic suspension its *transmissibility* must be known. This is best defined as the ratio of the disturbance produced in the foundation when the spring suspension is used to that when the vibrating body is rigidly attached, usually specified by the Greek letter ϵ (epsilon). C. R. Soderberg, of the Westinghouse Electric and Manufacturing Company, was the first to correctly formulate the fundamentals of this subject (reference 16). For purposes of analysis, ϵ is defined as the ratio of the vibration amplitude produced in the foundation when using the elastic suspension to that produced with a rigid connection. In many simple cases the foundation is relatively rigid and therefore its motion is comparatively small, so that the amplitude produced is proportional to the force of reaction. The value of ϵ can then be found easily as the ratio of the force transmitted to the foundation through the elastic suspension to that transmitted through a rigid suspension.

Suppose that the mass m supported by the spring k is acted upon by a periodic force of amplitude F as shown in Fig. 60. Then using the method of mechanical impedance,

$$y_1(k - m\omega^2) = F$$

$$y_1 = \frac{F}{k - m\omega^2}$$

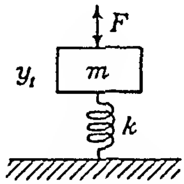


FIG. 60.—Elastic Cushion for Vibrating Mass.

and the force transmitted to the foundation through the spring is

$$ky_1 = \frac{kF}{k - m\omega^2}$$

That transmitted with k replaced by a rigid support is simply F. Therefore

$$\epsilon = \frac{kF}{k - m\omega^2} \div F = \frac{k}{k - m\omega^2}$$

$$\epsilon = \frac{\frac{k}{m}}{\frac{k}{m} - \omega^2} = \frac{\omega_c^2}{\omega_c^2 - \omega^2} = \frac{f_c^2}{f_c^2 - f^2} \quad . \quad (140)$$

where f_c is the free vibration frequency of the mass m on the spring k.

If $f/f_c = r$,

$$\epsilon = \frac{1}{1 - r^2} \quad . \quad . \quad . \quad . \quad . \quad (141)$$

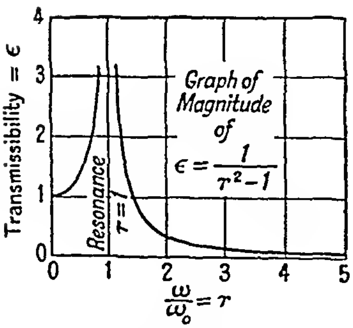


FIG. 61.—Vibration Transmissibility Curve.

Figure 61 shows how ϵ varies with f as given by equation (141) for a given resonant frequency f_c, ϵ being always counted positive, since the negative sign which appears when f/f_c is greater than unity merely signifies a phase opposition between the two forces, which does not concern us. It is evident that, for any reduction in trans-

missibility at all over that produced by rigid suspension, ϵ must be less than unity. Experience shows that, in ordinary cases of sound insulation, for an elastic suspension to be really effective, ϵ must be at least as small as $\frac{1}{30}$ to $\frac{1}{40}$. From Fig. 61 it is seen that if the spring suspension is so stiff that it resonates at a frequency such that f_c is higher than the exciting frequency so that r is less than unity, ϵ is always greater than unity and the suspension is worse than useless. It becomes effective only when f_c is $\frac{1}{6}$ to $\frac{1}{7}$ of f, that is, when the spring suspension is comparatively limber. For example, if $f/f_c = 7$,

$$\epsilon = \frac{1}{1 - 49} = \frac{1}{48} \text{ (dropping the negative sign)}$$

which gives good cushioning.

For the case of a torsional vibration the same analysis applies, giving the same formula for transmissibility, the elastic support being of course designed to cushion a torsional vibration in such a case, the symbols of equation (140) referring to torsional frequencies.

68. Special Case.—An elastic support may have more than one degree of freedom, each having its own natural frequency, f_c. Each of these values of f_c must be sufficiently below the exciting force f to insure sufficient cushioning. If one of these frequencies happens to be near f that particular mode of vibration is very likely to be picked up to some extent even though the exciting force f acts in a different direction from the line of displacement of that vibration. For instance, the present General Electric refrigerator pump mechanism is mounted on a three-spring suspension as indicated in Fig. 3. This has four definite independent frequencies to which it readily responds: (1) torsional, (2) sideways tilt, (3) up-and-down bounce, (4) sideways shearing oscillation. The last is highest of all and nearest the pump frequency f. All values of f_c, however, are far below the 120 ~ motor hum, which is thus

well isolated. The pump frequency comes through some as a slight quiver which is not objectionable because it is noiseless.

69. Effect of Damping in Spring Supports.—Consider a case like that of § 67 except that the spring support contains a small amount of damping.* Figure 62 illustrates this case where the damping factor b acts in parallel with the elastic constant as though it were contained in the spring material. From the rules of § 31,

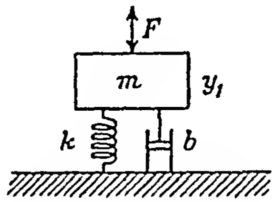


FIG. 62.—Elastic Suspension with Damping.

$$y_1(k - m\omega^2 + jb\omega) = F$$

$$y_1 = \frac{F}{k - m\omega^2 + jb\omega}$$

$$\epsilon = \frac{\text{force on base}}{F} = \frac{y_1(k + jb\omega)}{F} = \frac{k + jb\omega}{k - m\omega^2 + jb\omega}$$

Thus, expressing ϵ as a magnitude and dropping the phase angle,

$$\epsilon = \sqrt{\frac{k^2 + b^2\omega^2}{(k - m\omega^2)^2 + b^2\omega^2}} = \sqrt{\frac{\omega_c^4 + \dfrac{b^2\omega^2}{m^2}}{(\omega_c^2 - \omega^2)^2 + b^2\dfrac{\omega^2}{m^2}}} \quad (142)$$

since $\omega_c^2 = k/m$. This can be put in terms of frequencies, namely,

$$\epsilon = \sqrt{\frac{f_c^4 + \dfrac{b^2 f^2}{4\pi^2 m^2}}{(f_c^2 - f^2)^2 + b^2\dfrac{f^2}{4\pi^2 m^2}}} \quad . \quad . \quad . \quad (143)$$

This expression reduces to (140) when $b = 0$ as of course it should.

* See reference 4.

Note that damping slightly *increases* transmissibility. The increase is small, even when rubber is used as an elastic cushion, although rubber contains considerable damping. Thus, contrary to the belief of many engineers, damping in an elastic suspension may increase vibration transmission. Furthermore, springs do not damp a vibration, but prevent vibration transmission. Damping implies energy dissipation in the form of heat, and springs do not dissipate energy. Dissipation is effective, however, in reducing the amplitude of a resonant vibration in which the amplitude builds up to a peak value. Vibration ordinarily is transmitted to the base of a machine as a forced vibration. If the foundation were truly resonant, successful operation would be almost impossible.

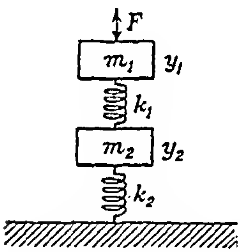


FIG. 63.—Elastic Suspension on Elastic Floor.

70. Effect of Foundation Characteristics on Transmissibility.—Important cases arise where the mass and elasticity of the foundation itself must be considered. The mechanics of this case is represented in Fig. 63. The transmissibility is defined as before as the ratio of the amplitude of vibration transmitted to the foundation through the spring support k to that with k replaced by a rigid support. In the simple case of § 67 the foundation displacement was taken as proportional to the force because of its comparative rigidity. This cannot be done, however, where the foundation itself is relatively yielding and may introduce an additional resonant frequency. We have now to deal with a compound system. The floor of a building on which an elastically supported machine is mounted may constitute such a system, m_2 and k_2 representing the equivalent mass and stiffness of the floor. In this case, ϵ must be calculated in terms of foundation vibration amplitudes rather than transmitted forces, with and without the elastic suspension k_1.

From the rules of § 31

$$\left.\begin{aligned} y_1(k_1 - m_1\omega^2) - y_2 k_1 &= F \\ y_2(k_1 + k_2 - m_2\omega^2) - y_1 k_1 &= 0 \end{aligned}\right\} \quad . \quad . \quad (144)$$

Solving for y_2, the foundation amplitude using the elastic suspension of stiffness k_1,

$$y_2 = \frac{Fk_1}{k_1k_2 - (k_1m_1 + k_1m_2 + k_2m_1)\omega^2 + m_1m_2\omega^4} \quad (145)$$

When k_1 becomes relatively very large, as with no elastic support between the vibrating body m_1 and floor m_2, y_2 assumes the value

$$y'_2 = \frac{F}{k_2 - (m_1 + m_2)\,\omega^2} \quad . \quad . \quad . \quad (146)$$

By definition, the transmissibility $= \dfrac{y_2}{y'_2} = \epsilon$

$$\epsilon = \frac{k_1k_2 - k_1(m_1 + m_2)\omega^2}{k_1k_2 - (k_1m_1 + k_1m_2 + k_2m_1)\omega^2 + m_1m_2\omega^4} \quad . \quad (147)$$

Dividing the numerator and denominator each by k_1k_2,

$$\epsilon = \frac{1 - \dfrac{m_1 + m_2}{k_2}\,\omega^2}{1 - \left(\dfrac{m_1}{k_2} + \dfrac{m_2}{k_2} + \dfrac{m_1}{k_1}\right)\omega^2 + \dfrac{m_1m_2}{k_1k_2}\,\omega^4} \quad . \quad . \quad (148)$$

When the floor stiffness k_2 becomes very large, this expression reduces to

$$\epsilon = \frac{1}{1 - \dfrac{m_1}{k_1}\,\omega^2} = \frac{1}{1 - \dfrac{\omega^2}{\omega_c^2}} = \frac{\omega_c^2}{\omega_c^2 - \omega^2} \quad . \quad . \quad . \quad (149)$$

where $\omega_c = \sqrt{k_1/m_1}$. The transmissibility is thus seen to reduce to the form given by equation (140) derived previously for the case of comparatively rigid foundation, as it should when k_2 is relatively large.

Expression (148) can be put in more convenient form if the two roots of the denominator are represented by ω_1^2 and ω_2^2 and $\dfrac{k^2}{m_1 + m_2} = \omega_3^2$. In this case

$$\epsilon = \frac{\dfrac{\omega_3^2 - \omega^2}{\omega_3^2}}{\left(\dfrac{\omega_1^2 - \omega^2}{\omega_1^2}\right)\left(\dfrac{\omega_2^2 - \omega^2}{\omega_2^2}\right)} \quad . \quad . \quad . \quad (150)$$

See reference 4.

The constant ω_3 and one of the roots of the denominator, ω_1 say, are both much larger than the other root ω_2 in ordinary cases, and with a fairly stiff floor where ω_3 and ω_1 are large compared with ω, the root ω_2 approaches $\omega_c = \sqrt{k_1/m_1}$, so that expression (150) becomes

$$\epsilon = \frac{1}{\dfrac{\omega_c^2 - \omega^2}{\omega_c^2}} = \frac{\omega_c^2}{\omega_c^2 - \omega^2}$$

which is identical to (149) just preceding. Note that ω_1 and ω_2 are the real resonant frequencies of the compound system (in radians per second), whereas ω_3 is not a real frequency but is the resonant frequency which the system would have if k_1 were a rigid connection.

71. Application to Single-phase Electric Motors.—In all single-phase motors the energy input must necessarily go through zero twice per cycle of the applied alternating electromotive force. This results in a double-frequency pulsating torque characteristic of all single-phase motors (reference 18), regardless of design except where special provision is made for energy storage as in the case of the capacitor type of motor. For small motors the armature current and consequently the periodic torque are relatively large at no load or small load. This torque rises and falls through an amplitude approximating the full-load torque in magnitude and at the magnetization frequency which is twice the alternating-current frequency. Thus

small, 60-cycle, single-phase motors when not elastically mounted *correctly* produce a 120-cycle hum which may be very objectionable. The hum from the average commercial washing machine may be transmitted all through the house where it is used. A correctly designed elastic mounting which will eliminate this hum can be applied at small cost.

Figure 64 shows a mounting of theoretically correct design. Since the periodic torque responsible for the hum has a frequency of 120 cycles, tending to oscillate the stator about its axis at this frequency, the whole motor must be mounted so that it will oscillate about this axis at a natural frequency at least as low as $\frac{1}{6}$ of $120 = 20 \sim$ per second. Each of the end plates of the motor is connected to the base through a rubber ring fixed to an inner and an outer brass ring. The inner brass ring is attached to a boss on the end plate surrounding the shaft; the outer brass ring is fixed in a hole in a riser from the base plate. The result is that torque on the motor causes the rubber layer between the brass rings to shear a small amount. The shearing elasticity of this ring is such that the free torsional oscillation frequency of the motor so mounted is about 20 cycles. This type of mounting has the advantage of being very resistant to sideways displacement as produced by belt pull, and at the same time cuts out the $120 \sim$ hum due to periodic torque.

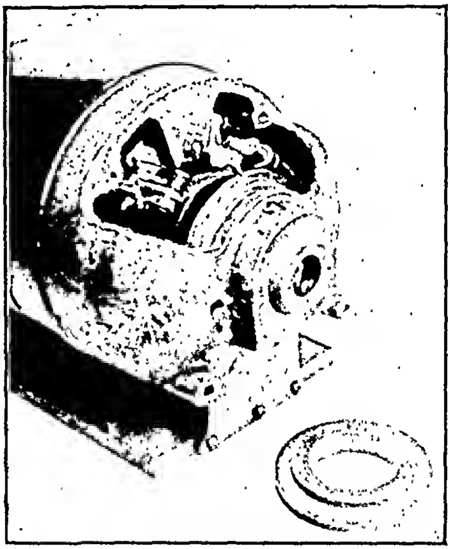


FIG. 64.—Motor with Elastic Suspension.

Other forms of elastic suspension can be successfully used for single-phase motors, but for best results the elas-

tic material should be placed *as near to the shaft as is practicable.* If this is not done, and the mounting is made rigid enough to withstand belt pull, its torsional stiffness is very likely to be too great to be consistent with the torsional moment of inertia of the stator in producing a low natural frequency. This greater stiffness is due to the longer moment arm when the support is at a greater distance from the stator axis. A support designed without calculating or measuring the particular natural frequency required will often afford little benefit. The majority of such elastic supports which the author has examined have been found to have too great torsional stiffness for good isolation.

From § 40 it will be recalled that a torque acting on a mass free in space causes it to rotate about its center of gravity regardless of where the torque is applied. Thus a single-phase motor tends to oscillate about its center of gravity which *usually* is about on the stator axis. If the motor has other masses attached to it, its center of gravity may be considerably displaced. The elastic support should take account of the consequent displacement of its natural axis of oscillation in such cases, and be designed so that the axis of elastic torsional motion coincides with the natural axis of oscillation. If this is not done, *transverse* forces will be transmitted through the support which the support may be transversely too rigid to cushion.

72. Application in Office Buildings and Hotels.—Modern office buildings and hotels must not be subjected to hums transmitted from electrical machinery. It is, therefore, essential to design ventilating systems with great care to guard against noise from power-driven fans. For this reason it has been the custom to avoid the use of alternating-current motors in such applications because even the best alternating-current motors will hum somewhat. Three-phase motors can be used successfully, however, with suitable vibration-absorbing bases. If cork cushioning pads are used, as they often are, the motor

should be bolted to a heavy concrete base and the cork placed under this. If this is not done the cork pad may transmit considerable hum. To be effective, the cork supports should be sufficiently thick and carry enough load to be well compressed. Without sufficient elastic compression (§ 9), the natural frequency of the support will not be lowered sufficiently to make the suspension effective.

Suppose the motor has a 120-cycle hum, and also the upper harmonics of 240 and 360 cycles are present, which the best three-phase motors exhibit to some extent. An elastic suspension which gives an up-and-down frequency of $120 \div 6 = 20$ cycles should sufficiently screen the transmission of all these frequencies, provided that the five other natural frequencies of the suspension are so placed that they are not excited.

From § 9 the compression of the elastic support necessary to produce this frequency in an up-and-down direction can at once be calculated as follows:

$$f = 20 = \frac{3.13}{\sqrt{d}}; \quad d = \left(\frac{3.13}{20}\right)^2 = 0.0245 \text{ inch.} \qquad (151)$$

Thus a compression of about 25 mils will give the required frequency, which from (140) gives a transmissibility of $\frac{1}{35}$ for 120 cycles and a much smaller transmissibility for the higher harmonics. Cork, however, is not a perfectly elastic material, and tests show that it should be compressed 3 or 4 times as much as an ideal spring to produce the required frequency, or as much as $\frac{1}{10}$ inch in the case just cited. (See reference 4.) If this precaution is taken, experience shows that it gives reasonably good results, though with the passage of time it tends to lose its elasticity somewhat under load. Considering its poor elastic qualities from the test standpoint it is surprising how well it works in some applications.

From the standpoint of elastic suspension the purpose of the concrete mass is to insure sufficient compression of the cork base to give it a low enough up-and-down fre-

quency and at the same time to permit the use of enough cork to insure sufficient transverse rigidity to withstand belt pull. With a little practice a designer can easily determine the weight of the slab and thickness and area of the cork base necessary for the required degree of compression.

High-grade rubber is elastically much superior to cork, nor does it lose its resilience appreciably with time, as is commonly believed, unless cheap grades are used.

It is possible to design satisfactory rubber suspensions for fan motors and for other such applications which have the advantage of requiring no concrete mass.

Expert knowledge is necessary, however; otherwise the support may be worse than useless. An important precaution in this type of suspension is to keep the natural frequencies of the support away from the operating speed of the motor, to avoid resonance in the operating range. If this range is considerable the design problem is correspondingly difficult. Every such suspension has *six resoant frequencies:* three possible angular vibrations, one about each of three perpendicular axes; and three linear vibrations, one in line with each of these axes. In other words, there are six degrees of freedom, each of which has its own resonant frequency (reference 17).

For satisfactory results the elastic support should be designed by the motor manufacturer, with a knowledge of the operating range of the motor.

It is difficult to design good steel spring suspensions of this type. Such springs do not offer the flexibility of design that rubber does in obtaining the right elasticity for every degree of freedom, and at the same time the necessary rigidity to carry belt pull. Furthermore, the small damping in rubber is a safeguard against vibration in passing through resonance.

Figure 65 shows a motor in service mounted on a specially designed rubber suspension. The rubber pads, placed in the metal boxes which protect them, are so

shaped that they have the correct stiffness in every direction. For a further discussion of this subject see reference 17.

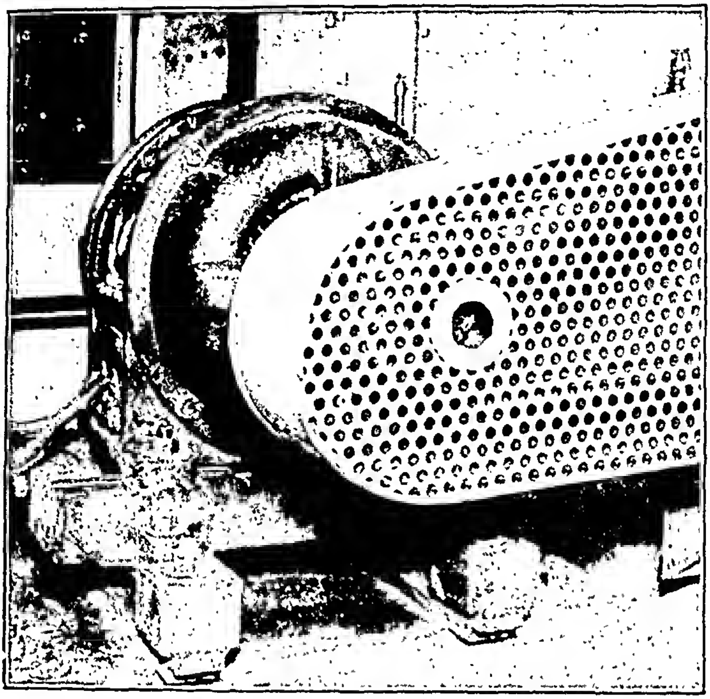


FIG. 65.—Elastic Suspension Installation.

73. Isolation from Vibration.—The reciprocal case from that of § 67 shown in Fig. 60 is illustrated in Fig. 66. Instead of finding how much motion or force is transmitted to the foundation with a given excitation of the mass m, the question here is to find out how much motion the mass m receives through the suspension k when the foundation vibrates through a fixed amplitude y_2.

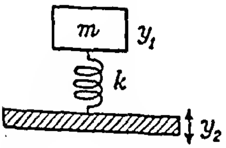


FIG. 66.—Vibration. Transmission to Mass.

The transmissibility is defined as the ratio of the amplitudes y_1 to y_2, or $\epsilon = y_1/y_2$. From § 31 the vector equation for the forces on m is given by

$$y_1(k - m\omega^2) - ky_2 = 0$$

Thus

$$\epsilon = \frac{y_1}{y_2} = \frac{k}{k - m\omega^2} = \frac{\omega_c^2}{\omega_c^2 - \omega^2} \quad . \quad . \quad (152)$$

which is exactly the same as equation (140) for the reciprocal case. The expression for ϵ when the spring contains damping is also the same for either case.

Isolation of this character is important where a piece of apparatus must be screened from the effects of a vibrating floor or building. Galvanometer suspensions and the balancing machine suspension taken up in § 63 are illustrations.

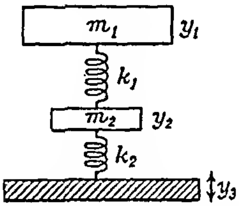


FIG 67.—Vibration Transmission to Compound System.

74. Vehicle Suspension.—An important practical application of this type of isolation is the spring suspension of vehicles such as automobiles.

Figure 67 shows a simplified diagram of the mechanical vibrating system of an automobile where the possibility of angular forward and backward pitching is omitted to simplify the problem. The elastic constant k_2 represents the resilience of the rubber tire, and k_1 the elastic constant of the springs of the vehicle. Setting down the vector equations according to the rules of § 31:

$$\begin{cases} y_1(k_1 - m_1\omega^2) - y_2k_1 = 0 \\ y_2(k_1 + k_2 - m_2\omega^2) - y_1k_1 - y_3k_2 = 0 \end{cases}$$

The transmissibility

$$\epsilon = \frac{y_1}{y_3} = \frac{1}{1 - \left(\frac{m_2}{k_2} + \frac{m_1}{k_1} + \frac{m_1}{k_2}\right)\omega^2 + \frac{m_1m_2}{k_1k_2}\omega^4} \quad (153)$$

This can be put in the form

$$\frac{y_1}{y_3} = \frac{1}{\left(\frac{\omega^2 - \omega_1^2}{\omega_1^2}\right)\left(\frac{\omega^2 - \omega_2^2}{\omega_2^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (154)$$

where ω_1^2 and ω_2^2 are the two roots of the denominator of equation (153) at which the system resonates and the transmissibility is indefinitely magnified. At points away from these resonances the transmissibility is small.

Note that y_1 is the motion of the vehicle and y_3 represents the amplitude imparted by bumps in the road. The higher frequency root, ω_1 say, corresponds to the resonance where m_1 and m_2 are vibrating towards and away from each other out of phase. "Washboard" roads causing the wheels to resonate up and down against the stops on the vehicle body illustrate this resonance. The lower resonance takes place when a car goes over much longer waves in the road, making the whole car body resonate up and down.

The damping produced by shock absorbers, not taken into account in the analysis presented, is essential in reducing the amplitudes of these two resonant frequencies, since every car must sometimes be subjected to one or the other of these two resonant frequencies on an uneven road. The function of shock absorbers is thus to damp resonant oscillations. So far as reducing shocks between resonant frequencies is concerned, they actually increase the transmissibility slightly, as explained for a simple case in § 69.

In general, the softer the springs the better, so far as is consistent with the stability of the car. This subject offers a field for further study. There is little doubt that, with careful study from the standpoint of these fundamentals, more can be accomplished in this direction than is realized by manufacturers.

CHAPTER XII

GENERAL THEORY OF VIBRATION DAMPING

75. Vibration Damping and Logarithmic Decrement.—The study of vibration damping is a whole field in itself which is of decided interest and importance in vibration work.

If a system of one degree of freedom with damping proportional to velocity, like that of § 28, vibrates freely with no stimulating force F, the sustained vibration term drops out and there remains only the expression

$$y = y_0 e^{-\alpha t} \cos \omega_0 t \quad . \quad . \quad . \quad . \quad . \quad (155)$$

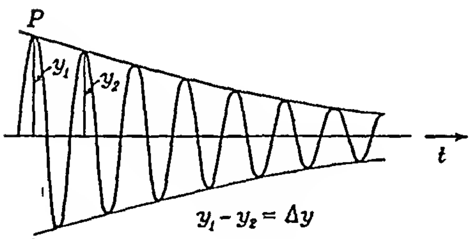


FIG. 68.—Decaying Vibration.

Figure 68 is a graphical representation of this equation. The best way to specify the damping in a vibrating system is by means of its *logarithmic decrement* δ, often spoken of simply as *decrement.*

The logarithmic decrement is defined as the natural logarithm of the ratio of two successive vibration amplitudes during the free vibration. At P in the vibration decrement curve shown, $\delta = \log (y_1/y_2)$.

From (155), when $t = 0$, $y_1 = y_0$.

One cycle later, $t = T$, the period, and $y_2 = y_0 e^{-\alpha T}$.

$$\delta = \log \frac{y_1}{y_2} = \log e^{\alpha T} = \alpha T = \frac{\alpha}{f_0} \quad . \quad . \quad (156)$$

where f_0 = natural frequency = $1/T$.

In this case, since both α and f_0 are constants which are independent of amplitude, δ = a constant at all times during the vibration decay. Thus a free vibration whose amplitude falls off exponentially with time like that given by equation (155) has a constant logarithmic decrement.

In practice, δ is seldom a constant, but either falls off or increases as the amplitude of the free vibration decreases. A velocity damping, however, where the frictional force is proportional to velocity (friction force $= b\,dy/dt$) always yields an exponential amplitude decrement curve like that of equation (155) with a constant logarithmic decrement.

Decaying electrical oscillations also fall off exponentially since electrical resistance acts as a velocity damping. The damping force in such an electrical circuit is a counter voltage of magnitude $R(dq/dt)$, where R = electrical resistance analogous to the friction constant b, and q is the charge displacement analogous to the mechanical displacement y.

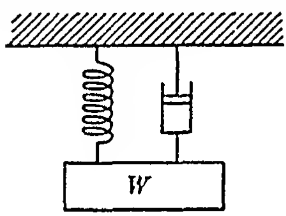


Fig. 69.—Simple Damped Vibration.

Although a velocity damping produces a logarithmic amplitude decrement of a free vibration this type of friction is not the only kind which will do this. The condition which has to be fulfilled for logarithmic amplitude decrement is simply that the dissipation of energy per cycle be proportional to amplitude squared, which condition velocity damping satisfies. This condition is also satisfied by a damping proportional to amplitude squared, which is independent of frequency.

76. Dissipation per Cycle and Loop Area.—The presence of damping in the elastic support of a weight W as shown in Fig. 69 causes the load deflection curve to depart from the ideal straight line AB, Fig. 70, and assume a curve which describes the loop $ACBD$ for a complete cycle.

Just as in the case of the magnetic hysteresis loop, the

area of this loop is a measure of the energy dissipated during the cycle of motion. This is easily shown to be true from the principle of work. The loop is described in the direction shown by the arrows. The force is measured upwards from the horizontal line of zero force at the bottom of the diagram and rises and falls above the mean value f_0 during the cycle of motion. The motion is measured horizontally from the vertical coordinate axis which corresponds to the rest position, the downward extension of the spring corresponding to increasing force being taken positive, and upward relaxation being taken as negative displacement. Now

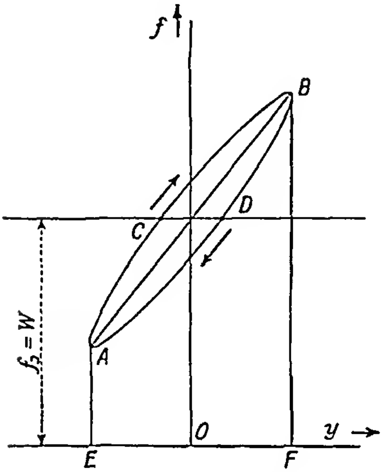


FIG. 70.—Stress-Strain Loop in Damped Vibration.

$$\text{Work} = \int f dy$$

(Note that f = *force* and not frequency as used here.) During the motion ACB this work input is positive since the work is done upon the spring and is equal to the area $EACBF$; during the motion BCA the work is supplied by the spring and is negative and equal to the area $EADBF$. Thus during each cycle the work input exceeds the output by the loop area. This loop area is thus equal to the dissipation per cycle in terms of the force and displacement units used. If f is in pounds and y in inches, the loop area measures the dissipation during the cycle in inch-pounds. The dissipation during one cycle will now be calculated. From § 31 the vector equation of forces is

$$y_1(k - m\omega^2 + jb\omega) = F \quad . \quad . \quad . \quad . \quad (157)$$

If $y = y_1 \sin \omega t$, all terms in the left-hand member are sine functions except the j term which is 90° ahead of the sine terms and is thus a cosine term. Thus, dropping the vector notation

$$y_1(k - m\omega^2) \sin \omega t + y_1 b\omega \cos \omega t = f \quad . \quad . \qquad (158)$$

where f is the instantaneous sinusoidal force, whose phase is determined by the left-hand member of the equation.

$$\text{Work per cycle} = \int f dy \text{ over a complete cycle}$$

$$\oint f dy = 2\int_0^\pi [y_1(k - m\omega^2) \sin \omega t + y_1 b\omega \cos \omega t] y_1 \cos \omega t \, d(\omega t)$$

since

$$dy = y_1 \cos \omega t d\,(\omega t)$$

Integrating over a cycle, the sin cos term is zero. Therefore

$$\oint f dy = 2\int_0^\pi y_1^2 b\omega \cos^2 \omega t \, d(\omega t) = \pi y_1^2 b\omega \qquad (159)$$

This is the work done per cycle by the force f and is therefore the dissipation per cycle due to the friction.

In this analysis, f is measured from the center of the loop instead of from the base line, thus leaving out the constant gravitational force component. The gravitational force f_0 yields no work in a complete loop which ends at the same point it started from since work done by gravitation due to an equal up-and-down motion is equal and opposite in sign, when the mass is brought back to its original position.

The expression $\pi y_1^2 b\omega$ equals the loop area in units of f and y. The loop can be shown to be a perfect ellipse for this type of damping. Its curve is given by the equations

$$\left.\begin{aligned} f &= y_1(k - m\omega^2) \sin \omega t + y_1 b\omega \cos \omega t \\ y &= y_1 \sin \omega t \end{aligned}\right\} \qquad (160)$$

Eliminating the parameter ωt by substituting, squaring, and using the relation $\sin^2 \omega t + \cos^2 \omega t = 1$, there is obtained after simplifying,

$$[(k - m\omega^2)^2 + b^2\omega^2]y^2 - 2(k - m\omega^2)\,yf + f^2 = y_1{}^2 b^2 \omega^2 \quad (161)$$

This is an equation of the second degree in y and f and must thus be a conic section, and since it is a closed curve it is an ellipse. The yf term is present because the ellipse is tilted with respect to the coordinate axes. At resonance, $k - m\omega^2 = 0$, and equation (161) becomes

$$\frac{y^2}{y_1{}^2} + \frac{f^2}{y_1{}^2 b^2 \omega^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad (162)$$

When the elastic force ky and mass reaction $-m\omega^2 y$ balance, as at resonance, the only remaining force is that due to damping, and the ellipse (Fig. 69) becomes horizontal, as seen from equation (162), with a semi major axis equal to the amplitude of deflection y_1 and the semi minor axis equal to the frictional force amplitude $y_1 b\omega$. The area of this ellipse in terms of its semi axes equals $\pi y_1{}^2 b\omega$, which checks equation (159).

Note that below resonance the ellipse twists counterclockwise from the horizontal position, and above resonance it twists clockwise. It is a twist in which every vertical element retains the same breadth but whose height depends upon ω. Thus the total area is proportional to frequency, as it must be with velocity damping.

When y and f are replaced by strain ϵ and stress σ at some point in an elastic medium supposed to contain a velocity damping force $\eta(d\epsilon/dt)$ corresponding to $b(dy/dt)$ of the previous problem, the loop area is given by an expression analogous to (159)

$$\text{Area} = \pi \epsilon_m{}^2 \eta \omega \quad . \quad . \quad . \quad . \quad . \quad (163)$$

where ϵ_m represents the amplitude of the sinusoidal strain ϵ, and η the coefficient of normal viscosity. In this case the dissipation is for unit volume of the medium per

strain cycle at the point under consideration. For another derivation of the expression see § 79.

77. Energy Expression for Decrement.—An important formula will now be derived which connects the logarithmic decrement δ with the dissipation per vibration cycle just discussed, which will be called ΔW, meaning a small amount of work.

Let W equal the total vibrational energy of some vibrating system vibrating in a given mode. It will be recalled from § 12 that, when a system vibrates in a given mode, the vibration of all of its particles is isochronous, that is, they vibrate all together in the same period, all reaching their maximum and minimum displacements at the same instant. Then $W = Ky^2$, where K is a constant depending on the system, the mode of vibration, and where y is measured. To be specific, take the cantilever shown in

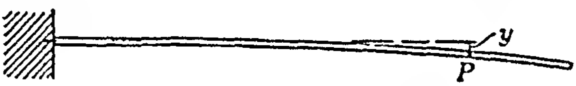


FIG. 71.—Vibrating Cantilever.

Fig. 71 vibrating in its gravest mode, and let y be measured at P. It is evident that K is different from the elastic constant k, and is a general proportionality constant. The expression $W = Ky^2$ says that the vibrational energy is proportional to the amplitude squared, the magnitude of K being such as to give the correct W for the case considered with y measured at P. This expression follows from Hooke's Law that stress is directly proportional to strain.

During a decaying vibration the amplitude at P gradually falls off. Assume that during one cycle it falls off from y to $y - \Delta y$. The dissipation per cycle ΔW must be the cause of the amplitude decrease Δy.

Since

$$W = Ky^2$$

$$W - \Delta W = K(y - \Delta y)^2$$

$$= K(y^2 - 2y\Delta y + \Delta y^2)$$

therefore $\Delta W = K \cdot 2y\Delta y$, neglecting $\Delta \overline{y}^2$, which is small compared with $2y\Delta y$.

Therefore

$$\frac{\Delta W}{2W} = \frac{K \cdot 2y\Delta y}{2Ky^2} = \frac{\Delta y}{y} \quad . \quad . \quad . \quad . \quad (164)$$

By definition (§ 75),

$$\delta = \log \frac{y + \Delta y}{y} = \frac{\Delta y}{y} \quad . \quad . \quad . \quad (165)$$

where the small terms in higher powers of $\Delta y/y$ are dropped because they are relatively small. This result is obtained by expanding the logarithm into its series.

Comparing (164) and (165) it is seen that

$$\frac{\Delta W}{2W} = \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

Thus δ can be expressed in three ways.

$$\left.\begin{array}{ll} (1) & \delta = \dfrac{\alpha}{f_0} \; (\S\ 75) \\[2ex] (2) & \delta = \dfrac{\Delta y}{y} \\[2ex] (3) & \delta = \dfrac{\Delta W}{2W} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (167)$$

The first is an exact expression.

The second and third are approximate, but the approximation is good for practical work where Δy is small compared with y and thus ΔW is small compared with W.

The second expression $\Delta y/y$ affords a quick method of estimating δ from the graph of a decaying vibration like that of Fig. 68.

The third expression is extremely useful in practical work in vibration damping as giving at once the energy dissipation produced when δ is known, or vice versa. If ΔW can be estimated for a damping device, δ can be immediately found, giving a quantitative estimate of the effectiveness of the damping means.

The great advantage of expressing the damping of a vibrating system in terms of δ is that δ is a dimensionless quantity, being a numerical ratio, and is thus the same whatever system of units be used to express energy and length. It is easily measured. It is readily found from the results of other investigators of vibration damping phenomena, however the results be set forth. By means of the energy expression when δ is known, ΔW is easily found, and vice versa.

The formulae (167) have great generality in that they apply at any instant, that is, at any point on an amplitude time decrement curve like that of Fig. 68. They apply to any system vibrating in any mode whose vibration decay can be represented by such a decrement curve. Therefore they apply regardless of how damping be produced, whether by viscous friction, externally or within the spring material, whether by coulomb friction or *solid friction*, and of course regardless of how the damping depends upon the amplitude of motion, or what be the shape of the "hysteresis" loop area during the vibration cycle. See reference 19.

78. Illustrations.—The application of the formulae for decrement just derived will be illustrated by a few simple cases.

(1) Comparing the first of equations (167) with the transient term of the solution of equation (44) of § 28,

$$\alpha = \frac{b}{2m} \qquad \ldots \ldots \qquad (168)$$

where $m = W/g$.

$$\delta = \frac{\alpha}{f_0} = \frac{b}{2mf_0} \qquad \text{But } f_0 = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ nearly}$$

Therefore

$$\delta = \frac{\pi b}{\sqrt{km}} \qquad \ldots \ldots \qquad (169)$$

which is a constant for this system.

Note that the vibrating system of § 27 for which this value of δ was found is the same as that analyzed in § 75 and shown in Fig. 69. For this system it was found that $\Delta W = \pi y^2 b\omega$ (dropping the subscript of y). Also for this system $W = \frac{1}{2}ky^2$. Note that in this expression K of § 77 $= \frac{1}{2}k$.

From the previous section $\delta = \Delta W/2W$.

Substituting the values of W and ΔW,

$$\delta = \frac{\pi b y^2 \omega}{ky^2} = \frac{\pi b \omega}{k} = \frac{\pi b}{\sqrt{km}} \quad . \quad . \quad (170)$$

since, as before, $\omega = \omega_c = \sqrt{k/m}$ very nearly, because δ is expressed in terms of ω for a free vibration. Note that (170) checks with (169) derived just previously.

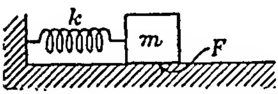


FIG. 72.—Vibration with Coulomb Friction Damping.

(2) Suppose a mass m to be vibrating back and forth and to be gradually brought to rest by sliding or "Coulomb" friction as indicated in Fig. 72. If the frictional force is F, considering a complete cycle,

$$\Delta W = 4Fy$$

also

$$W = \tfrac{1}{2}ky^2$$

Therefore

$$\delta = \frac{\Delta W}{2W} = \frac{4Fy}{ky^2} = \frac{4F}{ky} \quad . \quad . \quad . \quad (171)$$

Note that δ is not a constant but *increases* as y decreases.

In this case the amplitude falls off according to a straight line as shown in Fig. 73. This can be shown as follows.

For any vibration,

$$W = Ky^2 \ (\S\ 77)$$

$$\frac{dW}{dt} = 2Ky\frac{dy}{dt}$$

But $$\frac{dW}{dt} = -\Delta W \times \text{frequency} = -\Delta Wf$$

Substituting and solving for dy/dt,

$$\frac{dy}{dt} = \frac{-\Delta Wf}{2Ky} \qquad (172)$$

Thus if ΔW is linear in y, $dy/dt =$ constant; that is, the amplitude falls off with time according to a straight line.

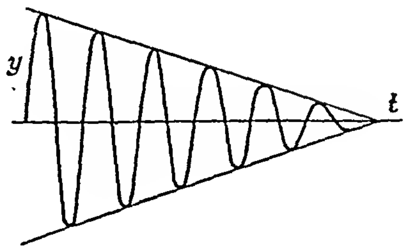


FIG. 73.—Decaying Vibration with Coulomb Friction Damping.

Equation (172) gives an exponential decrease of amplitude if ΔW depends on y^2 as previously noted, and if ΔW depends on y^3 the amplitude time decrement curve is hyperbolic, equation (172) giving the slope of the decrement curve in any case where ΔW can be expressed as a function of y.

79. Viscous Damping in Spring Material.—Suppose a mass m to be fixed to a reed which is rigidly attached at the lower end as shown in Fig. 74. Assume that all the friction present in the system causing vibration decay resides in the material of the reed itself and that this friction is viscous in character. Every elementary volume of the reed material goes through cyclical strains which for one vibration cycle are given by $\epsilon = \epsilon_m \sin \omega t$, neglecting the small falling off of amplitude in one cycle.

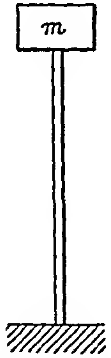


FIG. 74.—Vibrating Mass on Reed.

If the internal friction is of viscous character a resisting force is present which may be specified by the expression $f = \eta(d\epsilon/dt)$. This expression says that if unit volume of the material of the reed is strained with unit velocity, the frictional resistance $f = \eta$. The constant

η is the viscosity coefficient as defined by $f = \eta(d\epsilon/dt)$ and is perfectly analogous to b used in § 76 for the system of one degree of freedom. In this case a point in the solid material of the reed is considered, and is expanded to occupy unit volume by the imagination; in the case cited where b is used the whole system is considered as a unit.

As explained in § 76, if ΔW_1 = dissipation per *unit volume*

$$\Delta W_1 = \pi \epsilon_m{}^2 \eta \omega \qquad . \quad . \quad . \quad . \quad (173)$$

ΔW_1 is given the subscript because it represents dissipation per cycle per unit volume instead of for the whole vibrating system.

For the same unit volume the maximum strain energy stored in it per unit volume

$$W_1 = \tfrac{1}{2}\sigma_m \epsilon_m$$

Since $\sigma_m = E\epsilon_m$ (§ 14),

$$W_1 = \tfrac{1}{2}E\epsilon_m{}^2$$

where σ_m and ϵ_m are maximum values of stress and strain respectively at that point.

Therefore for unit volume

$$\delta = \frac{\Delta W_1}{2W_1} = \frac{\pi\eta\epsilon_m{}^2\omega}{E\epsilon_m{}^2} = \frac{\pi\eta\omega}{E} \qquad . \quad . \quad . \quad (174)$$

If this single unit cube were to vibrate by itself, (174) gives its decrement for viscous friction. To find δ for the whole reed, however, the value of ΔW and of W for the whole volume V of the reed must be used. For the entire reed material,

$$\delta = \frac{\Delta W}{2W} = \frac{\iiint \pi\eta\epsilon_m{}^2\omega dV}{\iiint E\epsilon_m{}^2 dV}$$

$$= \frac{\pi\eta\omega}{E}\frac{\iiint \epsilon_m{}^2 dV}{\iiint \epsilon_m{}^2 dV} = \frac{\pi\eta\omega}{E} \qquad . \quad . \quad . \quad (175)$$

which is the same as (174) for the single unit volume. It can be seen that this must be so because, for every unit volume, the ratio $\Delta W/2W$ is a constant for any type of vibration, for both depend upon ϵ_m^2. The integrals in (175) cancel off since they are identical.

The total vibrational energy lies in the reed as potential energy of bending at maximum deflection. All the dissipation also takes place in the reed. Every unit volume of the reed contributes dissipation and vibrational energy to the system in the same ratio, regardless of strain amplitude.

Note that, for the whole system vibrating, δ is much less than it would be for unit volume vibrating alone, for although δ is given by the same expression, ω_0 for unit volume is much higher than is ω_0 for the system as a whole. Although this is true of viscous damping, there are other types of damping where δ does not depend on ω and so is the same regardless of frequency.

It should be pointed out that δ is by definition based upon the natural frequency of the vibrating system considered and so is a physical constant of that system. Thus in the final result ω is always equal to ω_0, corresponding to the natural frequency f_0. In some of the previous cases ω_c has been used because of its close approximation to ω_0 (see § 27).

80. Solid Damping in an Elastic Material.—It has been shown beyond doubt that δ for practically all solid materials shows no such dependence upon frequency as given by (175). Common observation points toward this conclusion, since if this were not so the higher harmonics of a vibrating body, such as a bar, for example, would be immediately damped out because of their higher δ, leaving only the lower harmonics. A bell with a tone of comparatively high frequency would not ring at all, but act almost as dead as though made of putty. The viscous law of friction does not apply to metals nor to other solid materials as has been assumed by many investigators.

Experiments by the author * and by others show that

$$\Delta W_1 = \xi \sigma_m^2 = \xi E^2 \epsilon_m^2 \quad . \quad . \quad . \quad . \quad . \quad (176)$$

approximately when the stress amplitude σ_m is considerably below the elastic limit. This expression is accurate so far as independence of frequency is concerned. The dependence on ϵ_m^2 must be considered as a rather rough approximation. If σ_m approaches the elastic limit, ϵ_m usually has an exponent greater than 2. For practical work, however, where the vibrational stresses are mostly far below the elastic limit, expression (176) will suffice. This expression has an advantage from the standpoint of analysis that great simplification results compared with a law where ϵ_m is raised to a power other than 2.

Note particularly that expression (176) is independent of frequency. The constant ξ is merely a proportionality constant or solid friction constant, and depends upon the system of units employed as does E.

This type of friction has been called *solid friction* as being the type of internal friction characteristic of most solid materials.

For this law of dissipation,

$$\Delta W_1 = \xi E^2 \epsilon_m^2$$

and as in the previous section, for unit volume,

$$W_1 = \tfrac{1}{2} E \epsilon_m^2$$

therefore

$$\delta = \frac{\Delta W_1}{2W_1} = \frac{\xi E^2 \epsilon_m^2}{E \epsilon_m^2} = \xi E \quad . \quad . \quad . \quad . \quad (177)$$

It also can be shown as in the previous section that for the whole elastic mass

$$\delta = \xi E$$

In this case, however, every harmonic in the vibrating system has the same δ, regardless of its frequency. The higher harmonics die out most rapidly because they per-

* See reference 20.

form a larger number of vibrations in a given time than the lower ones although δ is the same for all harmonics. Here also, we know from common observation that the lower harmonics of a vibrating bar persist for a longer time than the higher harmonics, although the latter do show considerable persistence.

This is an important conclusion, since when δ is once measured for a given elastic material it is known for any vibrating system whose elastic energy and damping lie in that material, regardless of the frequency of the system.

The potential energy of a vibrating system is frequently partially or wholly derived from gravitation, a gravity pendulum being an example of the latter. In this case the damping friction which brings it to rest comes from pivot friction and air friction.

CHAPTER XIII

SHAFT WHIRLING DUE TO INTERNAL FRICTION AND TO OIL ACTION IN JOURNAL BEARINGS

81. Internal Friction and Shaft Whirling.—It seems rather surprising that internal friction within the material or structure of a revolving shaft or rotor should produce whirling. Ordinarily friction of this character in an elastic body causes vibrations to fall off and disappear. It is nevertheless true that friction in a rotor may cause it to build up a whirl under the right conditions.

Figure 75 (*a*) shows a vertical shaft carrying a flywheel deflected to the right, and (*b*) shows a section of the shaft at *CD* near its middle point. The shaft and flywheel are supposed to be whirling about the center line of the bearings *AB*, the shaft center at the section *CD* describing the circular path in the direction of the arrow, shown dotted in Fig. 75 (*b*). Since the elastic stiffness of the shaft balances the centrifugal force of the flywheel during the whirling motion, this whirl must take place at the angular velocity $\omega_c = \sqrt{k/m}$, where k is the elastic constant of the shaft at the point where the flywheel is attached, and m is the mass of the latter. The mass of the shaft is supposed to be small compared with that of the flywheel which it carries. Note that the shaft may revolve about its geometrical center S in either direction at any speed while the whirl takes place of fixed angular velocity ω_c. The latter motion may be compared to that of a planet about the sun, and the former to the revolution of the planet about its own axis. The two motions are independent of each other.

The whirl ω_c is resonant in character and is easily built up. If the whirl took place at any other speed it would

have to be forced and thus be exactly in step with some periodic exciting force. In this case no force is supposed to exist other than the steady disturbing force F_F produced by internal friction, which is always approximately tangent to the path of whirl. Since F_F acts in line with the direction of whirl, it does a definite amount of work per cycle in building up and sustaining the whirling motion. When the rate of energy dissipation just balances the rate of supply, the whirl will build up no further.

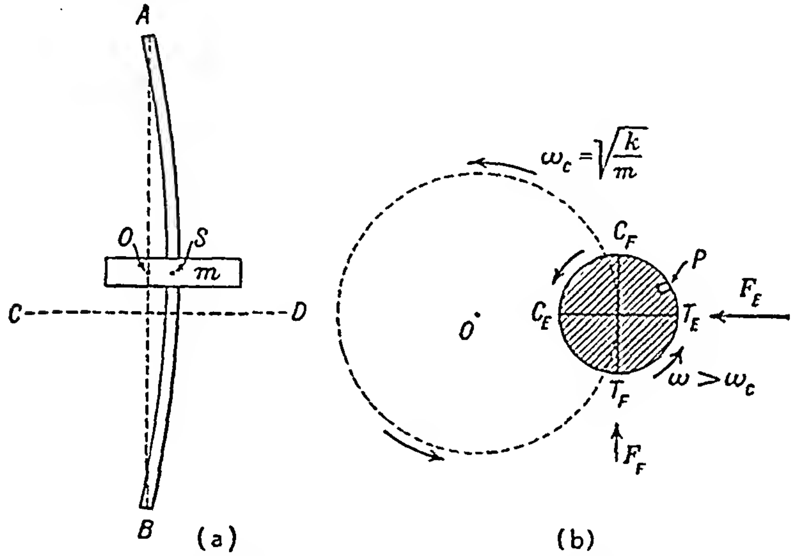


FIG. 75.—Shaft Whirl Excited by Internal Friction.

The force F_F is produced by internal friction within the rotor, and is in the positive direction of whirl only when the rotational speed of the shaft, ω, exceeds its whirling speed ω_c. If ω is less than ω_c the force F_F reverses and whirling is damped out.

Consider a section of the shaft as shown in Fig. 75 (*b*). The right-hand half of the section of the shaft is under elastic tension T_E. The left-hand half is under elastic compression C_E. As a fiber, whose section is shown at P, say, travels over the upper half of the section towards the region of elastic compression it is being shortened at a

certain rate. The internal friction present exerts a force of reaction to this shortening, so that a component of compressional force of small magnitude is exerted on this fiber. In like manner, a component of tension is required to overcome friction during elongation as the shaft fibers traverse the bottom half of the section. The resultant effect of this action is a small component of compression C_F in the upper half of the shaft section, and one of tension T_F in the lower half of the section, due to internal friction. The *elastic* tension and compression T_E and C_E due to bending produces a strong reaction F_E towards O, balancing the centrifugal force of the flywheel m due to the whirl ω_c. The frictional components T_F and C_F in like manner produce a much smaller reaction component F_F along the path of whirl, which can be balanced only by forces resisting the whirl. If these forces of resistance are relatively small, the whirl will build up.

This reasoning is general, and will apply to any kind of damping friction whatsoever in a rotor, whether it be in the shaft material or due to friction between parts of the rotor as its sides are alternately extended and compressed during its whirling motion as just described.

If the angular velocity of the shaft ω is the same as the whirling speed ω_c no friction will result as all the elements of the rotor remain strained the same amount without the cyclic change of strain depending on the difference between ω and ω_c. The shaft then would whirl with the same side out all the time, as though it had a permanent set. Thus for F_F to be positive the angular velocity of the shaft ω must exceed ω_c, the velocity of whirl. See reference 21.

82. Occurrence in Practice.—The chief danger from this type of whirling is that because of its rarity it is not recognized and understood in the few cases where it does occur. It is easy to produce such a whirl in a model shaft with a flywheel if one or two rings of axial length as great as the diameter of the shaft are shrunk on near the

middle of the span. Friction arises during deflection from a slight axial slipping of the shaft surface inside of the rings which is sufficient to cause a vigorous whirl when the shaft is run up above its critical speed. Tests show that the internal friction in the steel of a shaft is not sufficient to build up such a whirl. Built-up rotors which operate above the critical speed, however, must be watched, and should be so constructed to avoid as far as possible friction between parts when deflected. The impellers of compressors and disks of turbines should have narrow axial contacts with the shaft where they are shrunk on. Long, flexible laminated rotors which vibrate badly regardless of balance are open to suspicion. At least one case of a bad shaft failure probably due to this cause has come to the author's attention.

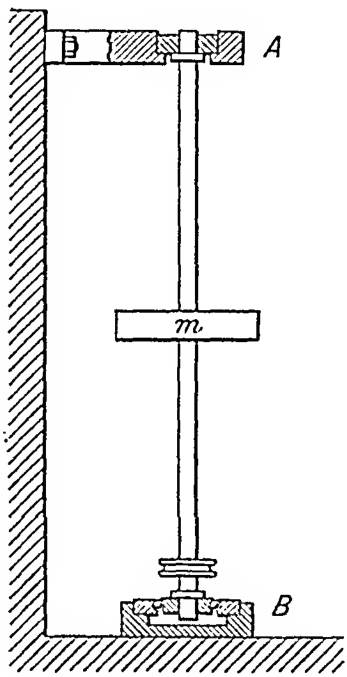


FIG. 76.—Shaft Whirl Excited by Oil Action.

83. The Oil Whip.—It has been found from practice that the action of oil in journal bearings is a more common cause of shaft whirling than internal friction in the rotor. This type of whirling will be taken up at this point because of its similarity to internal friction whirling, although it is not a damping or friction phenomenon.

If a small model shaft (Fig. 76) about 1 inch in diameter carrying a flywheel is placed on end with a ball bearing B at the bottom and a journal bearing A at the top with a small clearance which is well supplied with oil, it will be found, on slowly revolving the shaft, that the axis of the journal does not remain stationary, but moves in a small circle. This can be observed very well by machining the

point P on the upper end of the shaft, and noting its motion under a microscope while illuminated. The journal describes a circular motion in the bearing of amplitude depending upon the clearance. Furthermore, it will be noted that the angular velocity of this circular motion of the journal axis is exactly one-half of that of the shaft and flywheel. This motion is evidently produced by the action of the oil film, since the motion ceases when the oil supply is cut off. It can be explained by the action of the wedge-shaped oil film produced by the eccentric position of the journal. The oil film acts as a wedge and forces the journal around in the bearing in a forward direction at the speed with which the wedge of oil moves forward. This forward speed equals the average speed with which the oil wedge is dragged forward by the journal surface. One surface of the oil film is held stationary by the bearing while the other is dragged along at the journal speed, the average speed thus being about half the journal speed, just as the forward speed of the balls in a ball bearing is about half the journal speed.

For slow rotation the wedging action of the oil is not strong, and the journal performs the circular motion only when almost balanced on its end in the vertical position.

If the speed of the shaft is raised up to slightly above *twice* its critical speed it will then begin to whip, provided the bearings are both solidly supported so as to avoid the dissipation of vibrational energy. This building up of a whip is evidently due to the angular velocity of the wedge of oil falling in step with the critical speed of the shaft, so that a resonant vibration is established. When the shaft once starts to resonate at its critical speed, the oil wedge is held down to this speed even if the rate of revolution of the shaft is much higher than twice its critical speed. The driving force back of the wedge of oil increases, but this merely serves to stimulate the resonant whip of the shaft more strongly, while the velocity of the oil wedge is

held down to the critical speed frequency at which whipping takes place.

The tendency for this type of whipping is quite strong at rotational speeds of 25 per cent or more above twice the critical, so that a rotor in the horizontal position may be subject to this type of whipping, in spite of the force of gravitation which tends to hold the journal down on one side of the bearing. As soon as an oscillation is started at the resonant frequency of the rotor, the building-up action of the oil wedge becomes active.

That oil action is the cause of such whipping is shown by either cutting off the oil supply or reducing the speed of the rotor below twice the critical, since in either case the whipping ceases at once. The phenomenon is easily shown on a small model.

It has been found that increasing the bearing pressure by decreasing its area decreases the whipping tendency from oil action, but this is not a certain remedy. It is interesting that twisting the bearing axis out of line from the journal axis stops the whipping almost invariably. This method is uncertain in practice, as such lack of alignment is difficult to maintain and is apt to be soon lost during operation.

Another way to stop such whipping is through the use of spring supports with friction damping, but such supports as have been found effective are expensive and require considerable attention. The best cure is to avoid running journal sets over twice their first critical speed. See reference 22.

CHAPTER XIV

MEASUREMENT OF DAMPING CONSTANTS OF SOLID MATERIALS

84. Revolving Rod Method of Measuring Internal Friction.—One reason for discussing in § 81 the theory of shaft whipping as produced by internal friction is that the measurement of the disturbing force F_F gives a quick and direct method of measuring δ for the material of which the shaft is made.

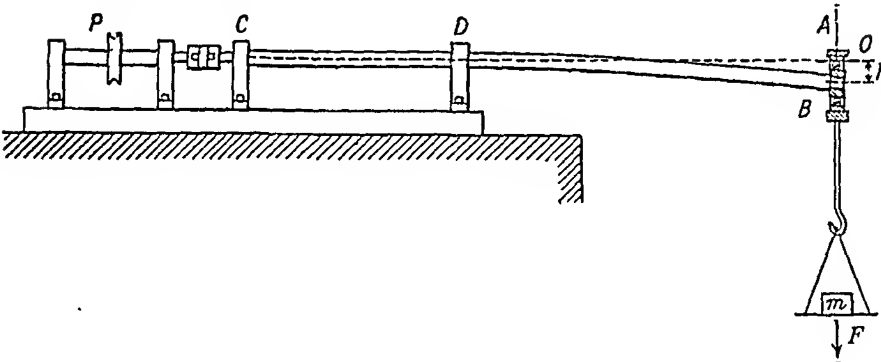


FIG. 77.—Revolving Rod Test for Internal Friction.

Instead of a vertical shaft, consider one which is horizontal, and for convenience, one which is overhanging as shown in Fig. 77.

If the overhanging shaft or rod CB is supported in the bearings C and D and driven through the pulley P, and at the same time its end is deflected downwards by the force F, produced by the weight and pan carried on another bearing at B, it will be found that, on revolving this deflected rod, it will also deflect a small amount sideways to the right or left, depending upon the direction of rota-

tion. Figure 78 shows a section at AB. The end of the shaft is deflected downwards by the amount h, and swings to the right through the angle ϕ. This angular deflection is caused by the internal friction in the shaft, and the magnitude of F_F (§ 81), which causes this deflection, is thus equal to $F \sin \phi$, while the elastic component $F_E = F \cos \phi$.

Let us apply the formula $\delta = \Delta W/2W$ to this system.

The torque required to produce the deflection ϕ against the gravitational force $f = F_F h = Fh \sin \phi$. Thus the total work done on the rod per cycle in overcoming its internal friction equals

$$\Delta W_R = 2\pi h F \sin \phi \quad . \quad . \quad . \quad . \quad . \quad (178)$$

where ΔW_R is the total internal friction work per cycle of revolution. Now if the gravitational force were eliminated and the mass m were vibrated up and down through the same amplitude h, it can be shown by an integration that

$$\Delta W = \frac{\Delta W_R}{2} \quad . \quad . \quad . \quad . \quad (179)$$

In words, expression (179) says that half as much energy is dissipated per cycle in a flexural vibration as is dissipated per cycle in revolving the rod while deflected, the amplitude h being the same in each case.

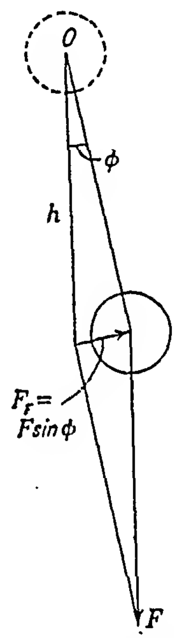


FIG. 78.—Transverse Displacement of End of Rod.

This follows from the law of solid friction given by (176), § 80, where dissipation depends upon stress squared, and may be shown as follows.

For the revolving rod, see Fig. 79, the fiber at P, of cross-sectional area $r d\theta dr$, goes through a stress cycle proportional to r while for the same rod with m vibrating up and down the stress cycle of this fiber is proportional to $r \sin \theta$, where AB is the neutral axis of the cross-section

for the vibration. Thus on the basis of § 80 the dissipation per cycle per unit length of fiber at P is $kr^2 \times rd\theta dr$ for the revolving rod and $kr^2 \sin^2 \theta \times rd\theta dr$ for the vibrating rod, where k is a proportionality constant, the same in each case, since h, Fig. 78, is taken the same for rotation as for vibration.

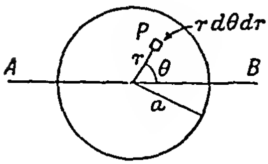


FIG. 79.—Cyclical Strains in Vibrating and Rotating Rod.

For unit length of rod, these expressions are integrated over the area of cross-section and will be found to differ by the factor 2. Assuming that the deflection curve of the rod is the same when it vibrates as when deflected downwards during rotation, the dissipation per cycle for the entire rod differs by the factor 2 for the two cases so that

$$\Delta W_R = 2\Delta W$$

Therefore from equation (179)

$$\Delta W = \pi h F \sin \phi$$

The potential energy W due to the elastic force component $F \cos \phi$ (Fig. 78) $= \frac{1}{2}hF \cos \phi$. Therefore

$$\delta = \frac{\Delta W}{2W} = \frac{\pi h F \sin \phi}{hF \cos \phi} = \pi \tan \phi \quad . \quad . \quad (180)$$

85. Damping Constants for Various Solids.—Thus by carefully measuring ϕ, δ can at once be found from equation (180). Its value is approximate except for the case where ΔW depends strictly on stress amplitude squared, in which case $\Delta W = \Delta W_R/2$ exactly.

It is easily established that δ and thus ΔW does not depend upon frequency since ϕ is entirely independent of the speed of rotation of the rod over a wide range, regardless of the load m on its end. By this method many metals and other materials have been found to exhibit this characteristic, so that it may be said that an outstanding characteristic of solid friction is that the dissipation per

vibration cycle is independent of frequency as given by (176) in § 80, all the way from less than one to many thousands of cycles per second.

By varying the force F the angle ϕ in general shows some variation, usually increasing with load. For a constant δ, ϕ must be constant also as is seen from expression (180).

In the following table is given a series of values of δ for different materials calculated by this method. These values are for the most part approximate as they depend on an average value of tan θ for the material in question except in special cases where tan ϕ is practically a constant for all loads and ΔW thus depends exactly on amplitude squared. Regardless of this approximation, these values of δ approximate the truth sufficiently to be very useful in practical work.

TABLE II

MATERIAL	TAN ϕ	δ
Rubber 90% pure *	....	0.26
Celluloid	.0144	.045
Tin, swaged	.0407	.129
Maple wood	.00669	.022
Zinc, swaged	.00630	.020
Glass	.00205	.0064
Aluminum, cold-rolled	.00107	.0034
Brass, cold-rolled	.00154	.0048
Copper, cold-rolled	.00159	.0050
Tungsten, swaged	.00524	.0165
Swedish iron, annealed	.00250	.0079
Phosphor bronze, annealed	.00101	.0032
Mild steel, cold-rolled	.00157	.0049
Molybdenum, swaged	.00219	.0069
Nickel, cold-rolled	.00102	.0032
Nickel steel, 3½% swaged	.00073	.0023
Monel, cold-rolled	.000454	.00143
Phosphor bronze, cold-rolled to maximum hardness	.000117	.00037

* Vibration test.

CHAPTER XV

VIBRATION DAMPING IN TURBINE BUCKETS

86. Vibration Amplitude from Energy Input.—Suppose that a system of one degree of freedom with the elastic constant k and mass m which is known to have a decrement δ is acted upon by a known pulsating magnetic force. It is required to find out the amplitude of vibration at resonance. A pulsating magnetic force can be very approximately represented by the expression

$$f = \frac{f_0}{2}(1 + \cos \omega t) \quad . \quad . \quad . \quad . \quad . \quad (181)$$

where f_0 is the *total* amplitude of the pulsating magnetic pull. Let the vibratory motion produced by this force be given by

$$y = y_0 \sin (\omega t + \alpha) \quad . \quad . \quad . \quad . \quad . \quad (182)$$

where α is the phase angle.

The work done per cycle equals

$$\Delta W = \oint f dy = 2\int_0^\pi \frac{f_0 y_0}{2}(1 + \cos \omega t) \cos (\omega t + \alpha)\, d(\omega t) \quad (183)$$

At resonance, dy is always positive while f is positive, and vice versa, so that $\alpha = 0$. Thus integrating (183) for $\alpha = 0$,

$$\Delta W = \frac{\pi f_0 y_0}{2}$$

But

$$\delta = \frac{\Delta W}{2W} = \frac{\pi f_0 y_0}{2ky_0^2} = \frac{\pi f_0}{2ky_0} \quad . \quad . \quad . \quad (184)$$

since

$$W = \tfrac{1}{2}ky_0^2$$

Solving for y_0,

$$y_0 = \frac{\pi f_0}{2k\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

which is the required relation.

This expression gives the maximum possible amplitude obtainable in a vibrating system with decrement δ from the pulsating force (184) acting upon the system at its resonant frequency.

87. Vibration Amplitude of Turbine Buckets.—It is sometimes required to find the approximate vibration amplitude of a reed-like body, such as a turbine bucket, which arises from a periodic force of the type given in the previous section, the damping constant δ being known approximately. The maximum periodic stress at the root of the bucket can then be calculated.

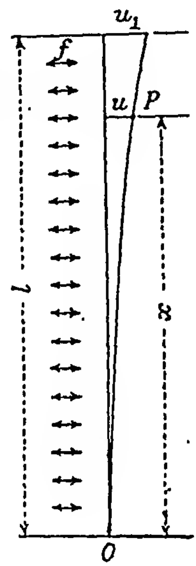


FIG. 80.—Turbine Bucket Vibration from Periodic Force.

Consider a bucket of length l.

At resonance the energy input per cycle equals the dissipation per cycle $= \Delta W$.

If x is measured along the bucket and u is the vibration amplitude at any point and f_0 is the maximum value per unit length of the pulsating force, $f = f_0/2\ (1 + \cos \omega t)$ distributed uniformly along the bucket, at some point P, the input per unit length is

$$\frac{d}{dx}\Delta W = \frac{\pi f_0 u}{2} \quad (\S 86)$$

Therefore the total input per cycle

$$\Delta W = \frac{\pi f_0}{2}\int_0^l u dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

The dissipation also $= \Delta W = 2W\delta$.

From Fig. 80

$$W = \int_0^l \frac{w}{2}\frac{\omega^2}{g} u^2 dx = \frac{w\omega^2}{2g}\int_0^l u^2 dx$$

where w = weight per unit length. Thus dissipation equals

$$2W\delta = \frac{w\omega^2\delta}{g}\int_0^l u^2 dx \quad . \quad . \quad . \quad (187)$$

Equating input to dissipation and rearranging,

$$\frac{\pi f_0 g}{2w\omega^2\delta} = \frac{\int_0^l u^2 dx}{\int_0^l u dx} = \alpha u_1 \quad . \quad . \quad . \quad (188)$$

where α = ratio of the integrals for the case where u_1, the amplitude of the tip of the bucket or of some other convenient point, is unity. The constant α is found from the vibration amplitude deflection curve, which depends upon the type of vibration. Since every element of the integral in the numerator is u times that in the denominator, and the amplitude of every element bears a constant ratio to u_1, the ratio of these integrals is a linear function of u_1 as shown. Thus

$$u_1 = \frac{\pi f_0 g}{2\alpha w\omega^2\delta} \quad . \quad . \quad . \quad . \quad . \quad (189)$$

To find the maximum stress σ_m at some point P of section modulus I where the bending moment is M, and where h is the distance from the neutral axis of bending to the extreme fiber, (189) is used in combination with the relation

$$\sigma_m = Eh\frac{d^2u}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad (190)$$

This relation is obtained from (20) of § 21, remembering that, when $y = h$, $\sigma = \sigma_m$ and that $1/\rho = d^2u/dx^2$. The term $(d^2u)/(dx^2)$ in (190) may then be expressed in terms of u_1 when the deflection curve is known, and (190) substituted in (189) in place of u_1, thus giving an expression for the required σ_m.

88. Application.—The use of this method will be shown by a relatively simple application:

Consider a turbine bucket of uniform cross-sectional area to be acted upon by a pulsating force of the type assumed, which acts in the plane of vibration as shown in Fig. 80, and which has a constant amplitude of f_0 per unit length of the bucket along its entire length. The bucket is assumed in this case not to be held at the tip by a shroud band, but to be vibrating at resonance with the exciting force as a simple cantilever beam.

FIG. 81.—Turbine Bucket Section.

Other constants are as follows:

Length $l = 5\frac{1}{8}$ inches.
I of cross-section $= 0.0037$ inch^4.
Distance h to neutral axis $= 0.25$ inch. (See Fig. 81.)
Decrement $\delta = 0.01$.

Find the maximum fiber stress σ_m at the root of the bucket. The amplitude deflection curve for this type of vibration is found from the general expression (91) of § 39 for flexural vibrations of a bar of uniform cross-section. By the use of the conditions (93) in this same section, the constants A, B, C, and D can be evaluated for the case of a fixed-free bar, using also the condition that $u = u_1$ when $x = l$.

Equation (91) then becomes

$$\left.\begin{aligned} u = u_1 \,(0.50128 \cosh m\,x - 0.36806 \sinh m\,x \\ -0.50128 \cos m\,x + 0.36806 \sin m\,x) \end{aligned}\right\} \quad (191)$$

where for a fixed-free bar $m = 1.875/l$ (§ 39). (This is the equation of the amplitude curve of Fig. 80.)

The constant α is found by graphical integrations using this deflection curve, and is approximately equal to 0.631.

The curvature of the bucket at its root when deflected according to equation (191) equals

$$\left(\frac{d^2u}{dx^2}\right)_{x=0} = 1.00256m^2u_1 \quad . \quad . \quad . \quad (192)$$

Thus from (190)

$$\sigma_m = 1.00256Ehm^2u_1 \quad . \quad . \quad . \quad . \quad (193)$$

Furthermore, for a fixed-free bar (§ 38),

$$\omega^2 = m^4\left(\frac{gEI}{w}\right) \quad . \quad . \quad . \quad . \quad . \quad (194)$$

Substituting (189) and (194) in (193),

$$\sigma_m = \frac{1.0026\pi f_0 h}{2m^2\alpha I\delta} \quad . \quad . \quad . \quad . \quad . \quad (195)$$

Since $m = \dfrac{1.875}{l}$ and $\alpha = 0.631$,

$$\sigma_m = 0.71\frac{f_0hl^2}{I\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

Substituting for h, l, I, and δ the values given at the beginning of the section:

$$\sigma_m = 120{,}000f_0$$

It is thus seen that if the total pulsating force ($f_0 \times l$) has an amplitude of 1 pound,

$$\sigma_m = \frac{120{,}000}{5\frac{1}{8}} = 23{,}400 \text{ pounds per inch}^2$$

This is dangerous from the viewpoint of fatigue, especially in the presence of steam.

When the bucket is held by a shroud the resonant force amplitude required to produce dangerous stresses (see Fig. 82) is considerably greater. This case can be handled by the same method, using the proper deflection curve,

derived from equation (91), and of course calculating a new value of α and expressing the integrals in terms of the deflection at some other point of the bucket than the tip. There are two zones of maximum stress, one at the section A, and the other at B, Fig. 82, for which the curvature may be found in terms of the same point on the bucket chosen for the evaluation of α.

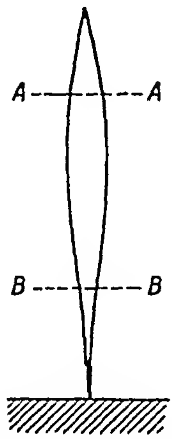


FIG. 82.—Turbine Bucket Vibration with Tip Supported.

Note from relation (195) that σ_m decreases directly as the decrement constant δ is increased.

This type of analysis tells us that short steam turbine buckets which occasionally fail require a pulsating force of considerable magnitude to break them, even when the bucket vibrates exactly in resonance with the force. This force is very likely to be the impulsive force on the bucket from passing equally spaced nozzle jets at high speed.

89. Dynamic Vibration Absorber.—If an objectionable vibration has great constancy of frequency, a powerful periodic force opposing this vibration may be obtained from a separate vibrating system which resonates to it.

Such a device is called a *dynamic vibration absorber*, and has been successfully used in some applications. It must be used with care, however, as a small change of frequency may cause it to aid instead of to oppose the objectionable vibration. (See reference 23.)

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